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ON ZEROS OF THE NUMERATOR AND DENOMINATOR POLYNOMIALS OF THIELE'S CONTINUED FRACTION

M. M. Pahirya

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We prove that the polynomials of canonical numerators and denominators of the interpolation and approximation convergents of Thiele's continued fractions have no common zeros. It is shown that the convergents of Thiele's continued fraction form a staircase sequence of normal Padé approximants. The region containing zeros of the denominator polynomial of the convergent of Thiele's continued fraction is also determined.

1. Introduction

A function of one real or complex variable can be interpolated by a polynomial, a spline, a Padé approximant, a continued fraction, etc.

It is known that the first works on the interpolation of functions by polynomials were written by Gregory and Newton as early as at the end of the 17th century. The subsequent development of the theory of interpolation of functions by polynomials was connected with the works by Waring, Lagrange, Euler, Chebyshev, Markov, Borel, Runge, Bernstein, Faber, Marcinkiewicz, and many other mathematicians.

For the first time, the problem of interpolation of functions by continued fractions was studied by Wrónski in 1811 and 1815–1817 [1, 2]. These works remained unknown for a long time due to the absence of references to Wrónski's results in the monographs by Thiele [3] and Nörlund [4], where the problem of interpolation was thoroughly investigated. The unique reference to Wrónski's works was made in the book [5] devoted to the history of continued fractions and Padé approximations.

Despite the fact that the analysis of Thiele interpolation continued fractions was given in the textbooks [6–8] and monographs [9–12], the number of works devoted to the interpolation by continued fractions and the methods of decomposition of functions in continued fractions is much smaller than the number of works dealing with the theory of approximation of functions by polynomials, splines, or Padé approximants. There are numerous unsolved problems in the theory of approximation by continued fractions. Some of these problems are connected solely with continued fractions.

In the present paper, we consider the problem of zeros of the canonical numerator and denominator of the Thiele interpolation continued fraction and Thiele continued fraction. In particular, we prove that the numerator and denominator polynomials do not have common zeros. We substantiate the fact that the convergents of the Thiele continued fraction obtained as a result of decomposition of a function in the Thiele continued fraction with the help of the Thiele formula form a staircase sequence of normal Padé approximants. We also determine the region of zeros of the denominator polynomial of convergent to the Thiele continued fraction.

2. Continued Fractions

We now present necessary facts from the theory of continued fractions.

Uzhhorod National University, Uzhhorod, Ukraine; Mukachevo State University, Mukachevo, Ukraine; e-mail: pahirya@gmail.com.

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Definition 1 [13]. *An infinite continued fraction is a triple* $\left[\{a_k\}_1^\infty,\ \{b_k\}_0^\infty,\ \{D_k\}_0^\infty\right]$ *of sequences, where the elements* $b_0,\ a_k,\ b_k,\ a_k \neq 0,\ k \in \mathbb{N}$, *are complex numbers and* $D_0,\ D_k,\ k \in \mathbb{N}$, *are elements of the extended complex plane* $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ *specified as follows: If a sequence of Möbius transformations*

$$s_0(w) := b_0 + w, \quad s_k(w) := a_k/(b_k + w), \quad k \in \mathbb{N},$$

is given, then

$$D_k := s_0 \circ s_1 \circ \ldots \circ s_k(0).$$

It follows from the definition that a finite continued fraction D_n has the form

$$D_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}},$$

which can be rewritten in the concise form as follows:

$$D_n = b_0 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots + \frac{a_n}{b_n}. \tag{1}$$

Similarly, an infinite continued fraction has the following concise form

$$D = b_0 + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots + \frac{a_k}{b_k +} \cdots. \tag{2}$$

The finite continued fraction (1) is called the nth convergent and the nth approximation of the continued fraction (2). A sequence of continued fractions $\{D_n\}$ is associated with sequences of complex numbers $\{P_n\}$ and $\{Q_n\}$ given by the following system of linear difference equations of the second order [9]:

$$Q_{-1} = 0, \qquad Q_0 = P_{-1} = 1, \qquad P_0 = b_0,$$

$$P_n = b_n P_{n-1} + a_n P_{n-2}, \qquad Q_n = b_n Q_{n-1} + a_n Q_{n-2}, \quad n \in \mathbb{N}.$$

The numbers P_n and Q_n are called, respectively, the nth canonical numerator and the nth canonical denominator of convergent (1), i.e., $D_n = P_n/Q_n$. The canonical numerators and denominators of the convergents D_n and D_{n-1} satisfy the determinant formula [9]

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} \prod_{i=1}^{n} a_i. \tag{3}$$

The continued fraction (1) can be also represented in the form of the ratio of two continuants.

Definition 2 [14]. *A determinant of the form*

$$\mathcal{H}_n^{\langle i\rangle} = \begin{vmatrix} b_i & a_{i+1} & 0 & 0 & \dots & 0 & 0 \\ -1 & b_{i+1} & a_{i+2} & 0 & \dots & 0 & 0 \\ 0 & -1 & b_{i+2} & a_{i+3} & \dots & 0 & 0 \\ 0 & 0 & -1 & b_{i+3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & b_{n-1} & a_n \\ 0 & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix}, \quad i=\overline{0,n}, \quad n \in \mathbb{N},$$

is called a continuant and can be represented in the following concise form:

$$\mathcal{H}_n^{\langle i\rangle} = \mathcal{K}\begin{pmatrix} & a_{i+1}, & a_{i+2}, & \dots, & a_{n-1}, & a_n \\ b_i, & b_{i+1}, & b_{i+2}, & \dots, & b_{n-1}, & b_n \end{pmatrix}.$$

It is known [15] that the relation

$$D_n = \frac{P_n}{Q_n} = \frac{\mathcal{H}_n^{\langle 0\rangle}}{\mathcal{H}_n^{\langle 1\rangle}} \tag{4}$$

is true. A continuant has the following property:

Theorem 1 [16]. *If an element a_k of the continuant $\mathcal{H}_n^{\langle i\rangle}$, where $i < k \le n$, is equal to zero and the other elements of $\mathcal{H}_n^{\langle i\rangle}$ are different from zero, then*

$$\mathcal{H}_n^{\langle i\rangle} = \mathcal{H}_n^{\langle k\rangle} \cdot \mathcal{H}_{k-1}^{\langle i\rangle}.$$

3. Zeros of the Numerator and Denominator Polynomials of the Thiele Interpolation Continued Fraction

Assume that a function f is given on a compact set $\mathcal{Z} \subset \mathbb{C}$. On the set of interpolation nodes $Z = \{z_i : z_i \in \mathcal{Z},\ z_i \ne z_j,\ i \ne j,\ i,j=\overline{0,n}\}$, the function takes values $w_i = f(z_i),\ i=\overline{0,n}$.

The function f on $\mathcal{Z}$ can be approximated by a Thiele interpolation continued fraction of the form [3, 4]

$$D_n(z) = \frac{P_n(z)}{Q_n(z)} = b_0 + \mathop{\mathrm{K}}_{i=1}^{n} \frac{z - z_{i-1}}{b_i}, \quad b_i \in \mathbb{C}, \quad i=\overline{0,n}. \tag{5}$$

The coefficients $b_i,\ i=\overline{0,n}$, of the Thiele interpolation continued fraction are determined either from the interpolation condition $D_n(z_i)=w_i$, where $i=\overline{0,n}$, or in terms of the inverted divided differences, or according to the recurrence relation in the form of a continued fraction [4, 17].

It is known that the numerator $P_n(z)$ and denominator $Q_n(z)$ of the Thiele interpolation continued fraction are polynomials whose degrees satisfy the inequalities $\deg P_n(z) \le [(n+1)/2]$ and $\deg Q_n(z) \le [n/2]$.

The polynomials $P_n(z)$ and $Q_n(z)$ are expressed in terms of the elements $b_0,\ b_i,\ z-z_{i-1},\ i=\overline{1,n}$, by the Euler–Minding formula [17, 18]:

$$P_n(z)=B_0^{[n]}\Bigg(1+\sum_{i=0}^{n-1}X_i(z)+\sum_{i_1=0}^{n-3}X_{i_1}(z)\sum_{i_2=i_1+2}^{n-1}X_{i_2}(z)+\sum_{i_1=0}^{n-5}X_{i_1}(z)\sum_{i_2=i_1+2}^{n-3}X_{i_2}(z)$$
$$\times\sum_{i_3=i_2+2}^{n-1}X_{i_3}(z)+\ldots+\sum_{i_1=0}^{n+1-2l}X_{i_1}(z)\sum_{i_2=i_1+2}^{n+3-2l}X_{i_2}(z)\ldots\sum_{i_l=i_{l-1}+2}^{n-1}X_{i_l}(z)\Bigg), \qquad (6)$$

$$Q_n(z)=B_1^{[n]}\Bigg(1+\sum_{i=1}^{n-1}X_i(z)+\sum_{i_1=1}^{n-3}X_{i_1}(z)\sum_{i_2=i_1+2}^{n-1}X_{i_2}(z)+\sum_{i_1=1}^{n-5}X_{i_1}(z)\sum_{i_2=i_1+2}^{n-3}X_{i_2}(z)$$
$$\times\sum_{i_3=i_2+2}^{n-1}X_{i_3}(z)+\ldots+\sum_{i_1=1}^{n+1-2m}X_{i_1}(z)\sum_{i_2=i_1+2}^{n+3-2m}X_{i_2}(z)\ldots\sum_{i_m=i_{m-1}+2}^{n-1}X_{i_m}(z)\Bigg), \qquad (7)$$

where

$$l=\left[\frac{n+1}{2}\right],\qquad m=\left[\frac{n}{2}\right],\qquad X_i=\frac{z-z_i}{b_i\,b_{i+1}},\quad i=\overline{0,n-1},\qquad B_k^{[n]}=\prod_{i=k}^{n}b_i,\quad k=0,1.$$

We introduce the continuant

$$\mathbf{T}_j^{\langle i\rangle}(z)=\mathcal{K}\begin{pmatrix} & z-z_i, & z-z_{i+1}, & \ldots, & z-z_{j-1}\\ b_i, & b_{i+1}, & b_{i+2}, & \ldots, & b_j\end{pmatrix},\quad i<j. \qquad (8)$$

Then the Thiele interpolation continued fraction (5) is expressed in the form (4) of the ratio of continuants (8) as follows:

$$D_n(z)=\frac{\mathbf{T}_n^{\langle 0\rangle}(z)}{\mathbf{T}_n^{\langle 1\rangle}(z)}.$$

Theorem 2. *If, for some value $n\in\mathbb{N}$, the coefficients of the Thiele interpolation continued fraction (5) are finite and not equal to zero and the function f takes nonzero values at the nodes, i.e., $f(z_i)\neq 0,\ i=\overline{0,n}$, then the polynomials of the numerator $P_n(z)$ and denominator $Q_n(z)$ do not have common zeros, i.e., $P_n(z)$ and $Q_n(z)$ are coprime polynomials over the field of complex numbers and $P_n(z)/Q_n(z)$ is an irreducible rational fraction.*

Proof. We proceed by induction. For $n=1$, the polynomials in the numerator $P_1(z)=b_0b_1+z-z_0$ and denominator $Q_1(z)=b_1$ do not have common zeroes. For $n=2$, we get

$$P_2(z)=b_0b_1b_2+b_2(z-z_0)+b_0(z-z_1)\qquad\text{and}\qquad Q_2(z)=b_1b_2+z-z_1.$$

In view of the determinant formula (3), we get

$$P_2(z)Q_1(z)-P_1(z)Q_2(z)=-(z-z_0)(z-z_1).$$

Since $P_1(z)$ and $Q_1(z)$ do not have common zeros, the nodes z_0 or z_1 can be common zeros of the polynomials $P_2(z)$ and $Q_2(z)$. It is easy to see that z_0 and z_1 are not common zeros of $P_2(z)$ and $Q_2(z)$.

Assume that, for $n = \overline{0, k-1}$, the polynomials $P_n(z)$ and $Q_n(z)$ do not have common zeroes. Thus, for $n = k$, it follows from the determinant formula that

$$P_k(z)Q_{k-1}(z) - P_{k-1}(z)Q_k(z) = (-1)^{k-1}(z-z_0)(z-z_1)\dots(z-z_{k-1}).$$

By the induction assumption, the polynomials $P_{k-1}(z)$ and $Q_{k-1}(z)$ do not have common zeros. Thus, only one interpolation node $z_0, z_1, \dots, z_{k-1}$ can be the common zero of the polynomials $P_k(z)$ and $Q_k(z)$. Let z_s, $0 \le s \le k-1$, be one of the nodes. Then the continuants $\mathbf{T}_k^{\langle 0\rangle}(z_s)$ and $\mathbf{T}_k^{\langle 1\rangle}(z_s)$ are equal to

$$\mathbf{T}_k^{\langle 0\rangle}(z_s) = \mathcal{K}\begin{pmatrix} & z_s - z_0, & \dots, & z_s - z_{s-1}, & 0, & z_s - z_{s+1}, & \dots, & z_s - z_{k-1} \\ b_0, & b_1, & \dots, & b_s, & b_{s+1}, & b_{s+2}, & \dots, & b_k \end{pmatrix},$$

$$\mathbf{T}_k^{\langle 1\rangle}(z_s) = \mathcal{K}\begin{pmatrix} & z_s - z_1, & \dots, & z_s - z_{s-1}, & 0, & z_s - z_{s+1}, & \dots, & z_s - z_{k-1} \\ b_1, & b_2, & \dots, & b_s, & b_{s+1}, & b_{s+2}, & \dots, & b_k \end{pmatrix}.$$

By Theorem 1, we obtain

$$\mathbf{T}_k^{\langle 0\rangle}(z_s) = \mathbf{T}_k^{\langle s\rangle}(z_s)\mathbf{T}_{s-1}^{\langle 0\rangle}(z_s) \quad \text{and} \quad \mathbf{T}_k^{\langle 1\rangle}(z_s) = \mathbf{T}_k^{\langle s\rangle}(z_s)\mathbf{T}_{s-1}^{\langle 1\rangle}(z_s).$$

This implies that

$$\frac{P_k(z_s)}{Q_k(z_s)} = \frac{P_{s-1}(z_s)}{Q_{s-1}(z_s)}.$$

By the the induction assumption, the polynomials $P_t(z)$ and $Q_t(z)$, $t = \overline{0, k-1}$, do not have common zeros. Thus, the node z_s is not a common zero of $P_k(z)$ and $Q_k(z)$. Since z_s is arbitrary, we conclude that the indicated polynomials do not have common zeros. Hence, the theorem remains true for $n = k$.

4. Zeros of the Thiele Continued Fraction

It is known [3, 4] that, in the limit case, the Thiele formula, which is an analog of the Taylor formula in the theory of continued fractions, can be obtained from the Thiele interpolation continued fraction (5). If the interpolation nodes $z_0, z_1, \dots, z_k \to z_*$, where $z_* \in \mathcal{Z}$, then the limit value $\rho_k[z_0, z_1, \dots, z_k; f]$ of the inverse difference of order k is called the inverse Thiele derivative of order k for the function f at the point z_* on the compact set $\mathcal{Z}$ and denoted by ${}^{(k)}f(z_*)$, i.e.,

$${}^{(k)}f(z_*) = \lim_{z_0, z_1, \dots, z_k \to z_*} \rho_k[z_0, z_1, \dots, z_k; f], \quad k \in \mathbb{N}.$$

The inverse Thiele derivatives are given by the following recurrence relations [3]:

$${}^{(k)}f(z_*) = k \cdot {}^{(1)}\big({}^{(k-1)}f(z_*)\big) + {}^{(k-2)}f(z_*), \quad k \in \mathbb{N}_2 = \mathbb{N}\backslash\{1\},$$

$${}^{(0)}f(z_*) = f(z_*), \quad {}^{(1)}f(z_*) = 1/f'(z_*).$$

If a function f has the inverse Thiele derivatives up to the nth order, inclusively, in a certain neighborhood of the point z_*, then, by the Thiele formula, it can be represented in the form

$$f(z) = b_0(z_*; f) + \frac{z - z_*}{b_1(z_*; f)} + \frac{z - z_*}{b_2(z_*; f)} + \ldots + \frac{z - z_*}{b_n(z_*; f)} + \frac{z - z_*}{R_n(z; f)},$$

where $R_n(z; f)$ is the remainder of the continued fraction. The coefficients $b_i(z_*; f)$ are expressed in terms of the inverse Thiele derivatives as follows:

$$b_0(z_*; f) = f(z_*), \quad b_1(z_*; f) = {}^{(1)}f(z_*),$$

$$b_n(z_*; f) = {}^{(n)}f(z_*) - {}^{(n-2)}f(z_*), \quad n \in \mathbb{N}_2.$$

If a function f has infinitely many nonzero inverse Thiele derivatives in a certain neighborhood of the point $z = z_*$, then we get the following decomposition of this function in a formal Thiele continued fraction:

$$f(z) = b_0(z_*; f) + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{z - z_*}{b_k(z_*; f)}. \tag{9}$$

The properties of the inverse Thiele derivatives, some examples of decompositions of the functions in continued fractions with the help of the Thiele formula, and the substantiation of the domains of convergence and uniform convergence of the obtained decompositions can be found in [3, 4, 10, 17, 19].

We now represent the convergent $D_n(z; z_*, f)$ of the Thiele continued fraction (9) in the form

$$D_n(z; z_*, f) = \frac{P_n(z; z_*, f)}{Q_n(z; z_*, f)} = b_0(z_*; f) + \mathop{\mathrm{K}}_{k=1}^{n} \frac{z - z_*}{b_k(z_*; f)}, \tag{10}$$

where the polynomials in the numerator $P_n(z; z_*, f)$ and in the denominator $Q_n(z; z_*, f)$ can be expressed via the elements $b_i(z_*; f)$, $i = \overline{0, n}, z - z_*$ of the Thiele continued fraction (10) by the Euler–Minding formula (6), (7) as follows:

$$\begin{aligned} P_n(z; z_*, f) = B_0^{[n]} \Bigg(& 1 + (z - z_*) \sum_{i=0}^{n-1} A_i + (z - z_*)^2 \sum_{i_1=0}^{n-3} A_{i_1} \sum_{i_2=i_1+2}^{n-1} A_{i_2} \\ & + (z - z_*)^3 \sum_{i_1=0}^{n-5} A_{i_1} \sum_{i_2=i_1+2}^{n-3} A_{i_2} \sum_{i_3=i_2+2}^{n-1} A_{i_3} + \ldots \\ & + (z - z_*)^l \sum_{i_1=0}^{n+1-2l} A_{i_1} \sum_{i_2=i_1+2}^{n+3-2l} A_{i_2} \ldots \sum_{i_l=i_{l-1}+2}^{n-1} A_{i_l} \Bigg), \end{aligned} \tag{11}$$

$$Q_n(z; z_*, f) = B_1^{[n]} \Bigg(1 + (z - z_*) \sum_{i=1}^{n-1} A_i + (z - z_*)^2 \sum_{i_1=1}^{n-3} A_{i_1} \sum_{i_2=i_1+2}^{n-1} A_{i_2}$$

$$+ (z-z_*)^3 \sum_{i_1=1}^{n-5} A_{i_1} \sum_{i_2=i_1+2}^{n-3} A_{i_2} \sum_{i_3=i_2+2}^{n-1} A_{i_3} + \ldots$$

$$+ (z-z_*)^m \sum_{i_1=1}^{n+1-2m} A_{i_1} \sum_{i_2=i_1+2}^{n+3-2m} A_{i_2} \ldots \sum_{i_m=i_{m-1}+2}^{n-1} A_{i_m} \Bigg), \tag{12}$$

where

$$A_i = \frac{1}{b_i(z_*; f)\, b_{i+1}(z_*; f)}, \quad i = \overline{0, n-1}, \qquad B_l^{[n]} = \prod_{i=l}^{n} b_i(z_*; f), \quad l = \left[\frac{n+1}{2}\right], \quad m = \left[\frac{n}{2}\right].$$

Theorem 3. *If the coefficients $b_k = b_k(z_*; f)$, $k = \overline{1, n}$, of convergent (10) are finite and nonzero and, moreover, $f(z_*) \neq 0$, then the canonical numerator $P_n(z; z_*, f)$ and the canonical denominator $Q_n(z; z_*, f)$ do not have common zeros, i.e., $P_n(z; z_*, f)$ and $Q_n(z; z_*, f)$ are coprime polynomials over the field of complex numbers and $P_n(z; z_*, f)/Q_n(z; z_*, f)$ is an irreducible rational function.*

Proof. We proceed by induction. For $n = 1$, we see that the polynomials $P_1(z; z_*, f) = b_0 b_1 + z - z_*$ and $Q_1(z; z_*, f) = b_1$ do not have common zeros. Further, if $n = 2$, then

$$P_2(z; z_*, f) = b_0 b_1 b_2 + (b_0 + b_2)(z - z_*)$$

and $Q_2(z; z_*, f) = b_1 b_2 + z - z_*$. According to the determinant formula (3), we find

$$P_2(z; z_*, f) Q_1(z; z_*, f) - P_1(z; z_*, f) Q_2(z; z_*, f) = -(z - z_*)^2.$$

Since the polynomials $P_1(z; z_*, f)$ and $Q_1(z; z_*, f)$ do not have common zeros, we conclude that only z_* can be a common zero of $P_2(z; z_*, f)$ and $Q_2(z; z_*, f)$. However, z_* is a zero neither for $P_2(z; z_*, f)$, nor for $Q_2(z; z_*, f)$. Thus, the theorem is true for $n = 1, 2$.

Assume that the theorem is true for $n = k - 1$. For $n = k$, the determinant formula takes the form

$$P_k(z; z_*, f) Q_{k-1}(z; z_*, f) - P_{k-1}(z; z_*, f) Q_k(z; z_*, f) = (-1)^{k-1} (z - z_*)^k.$$

By the induction assumption, $P_{k-1}(z; z_*, f)$ and $Q_{k-1}(z; z_*, f)$ do not have common zeros. Only z_* can be a common zero of $P_k(z; z_*, f)$ and $Q_k(z; z_*, f)$.

It follows from relations (11) and (12) that $P_k(z_*; z_*, f) = B_0^{[k]}$ and $Q_k(z_*; z_*, f) = B_1^{[k]}$. Thus, z_* is not a zero of polynomials in the numerator $P_k(z; z_*, f)$ and in the denominator $Q_k(z; z_*, f)$. Hence, the theorem remains true for $n = k$.

Remark 1. In the monograph [20], a similar statement was proved for the case of continued RIT-fractions.

5. Sequences of Padé Approximants for the Thiele Continued Fraction

It is known [21] that if a function f is given by a formal power series

$$f(z) = \sum_{i=0}^{\infty} c_i z^i, \tag{13}$$

then the Padé approximant $[L/M]_f$ of the function f is defined as an irreducible rational function

$$R^{[L/M]}(z) = P^{[L/M]}(z)/Q^{[L/M]}(z),$$

where $Q^{[L/M]}(0) = 1,$ satisfying the relation

$$R^{[L/M]}(z) = f(z) + O\left(z^{L+M+1}\right).$$

A two-dimensional array of rational functions $\left\{R^{[M/L]}(z),\ L, M \in \mathbb{N} \cup \{0\}\right\}$ is called the Padé table for the formal power series (13). A Padé approximant $[L, M]_f$ is called normal if

$$\deg P^{[L/M]}(z) = L \qquad \text{and} \qquad \deg Q^{[L/M]}(z) = M.$$

We rewrite (11) and (12) in the form

$$P_n(z; z_*, f)(z) = p_l(w) = w^l + \mathbf{a}_1 w^{l-1} + \ldots + \mathbf{a}_{l-s} w^s + \ldots + \mathbf{a}_{l-1} w + \mathbf{a}_l, \tag{14}$$

$$Q_n(z; z_*, f)(z) = q_m(w) = w^m + \mathbf{b}_1 w^{m-1} + \ldots + \mathbf{b}_{m-k} w^k + \ldots + \mathbf{b}_{m-1} w + \mathbf{b}_m, \tag{15}$$

where

$$\mathbf{a}_l = B_0^{[n]}, \qquad \mathbf{a}_{l-s} = B_0^{[n]} \sum_{i_1=0}^{n+1-2s} A_{i_1} \sum_{i_2=i_1+2}^{n+3-2s} A_{i_2} \ldots \sum_{i_s=i_{s-1}+2}^{n-1} A_{i_s}, \quad s = \overline{1, l-1}, \quad l = \left[\frac{n+1}{2}\right],$$

$$\mathbf{b}_m = B_1^{[n]}, \qquad \mathbf{b}_{m-k} = B_1^{[n]} \sum_{i_1=1}^{n+1-2k} A_{i_1} \sum_{i_2=i_1+2}^{n+3-2k} A_{i_2} \ldots \sum_{i_k=i_{k-1}+2}^{n-1} A_{i_k}, \quad k = \overline{1, m-1}, \quad m = \left[\frac{n}{2}\right], \tag{16}$$

$$w = z - z_*, \quad A_i = 1/b_i b_{i+1}.$$

Theorem 4. *If, for each value $n \in \mathbb{N}$, the coefficients $b_k,\ k = \overline{1, n}$, of the continued fraction (10) take finite nonzero values and $b_0 = f(z_*) \neq 0$, then the sequence of continued fractions $\{D_n(z; z_*, f)\}$ form a staircase sequence of normal Padé approximants*

$$\left\{R^{[0/0]}(w), R^{[1/0]}(w), R^{[1/1]}(w), R^{[2/1]}(w), R^{[2/2]}(w), \ldots\right\}$$

of the function f.

Proof. It follows from (14) and (15) that

$$D_n(z; z_*, f) = R^{[l/m]}(w).$$

By Theorem 3, the polynomials $p_l(w)$ and $q_m(w)$ do not have common zeros and, hence,

$$R^{[l/m]}(w) = p_l(w)/q_m(w)$$

is an irreducible rational function. It is easy to see that

$$q_m(0) = B_1^{[n]} \neq 0.$$

We divide the polynomials $p_l(w)$ and $q_m(w)$ by $B_1^{[n]}$. Hence, $R^{[l/m]}(w) = \bar{p}_l(w)/\bar{q}_m(w)$ is an irreducible function and $\bar{q}_m(0) = 1$. It follows from (14) and (15) that $\deg p_l(w) = l$ and $\deg q_m(w) = m$. In [22], it was proved that the Thiele continued fraction (9) also has the formal power series (13). Therefore,

$$R^{[l/m]}(z - z_*) = f(z) + O(z^{l+m+1}).$$

Thus, the sequence $\left\{R^{[l/m]}(w) = D_n(z; z_*, f)\right\}$ is a sequence of normal Padé approximants of the function f.

6. Domain of Zeros for the Denominator of the Thiele Continued Fraction

Consider a domain of the complex plane that contains all zeros of the canonical denominator $Q_n(z; z_*, f)$ of the Thiele continued fraction (10). We use the following statement:

Proposition 1 [23]. *Suppose that $p_n(z) = z^n + c_1 z^{n-1} + \ldots + c_{n-1} z + c_n$ is a polynomial with nonzero coefficients and that $\bar{z}_i,\ i = \overline{1, n}$, are zeros of this polynomial. Then*

$$\max_{1 \le i \le n} |\bar{z}_i| = r_0 \le \max\left\{2|c_1|,\ 2\left|\frac{c_2}{c_1}\right|,\ 2\left|\frac{c_3}{c_2}\right|, \ldots,\ 2\left|\frac{c_{n-1}}{c_{n-2}}\right|,\ \left|\frac{c_n}{c_{n-1}}\right|\right\}. \tag{17}$$

Theorem 5. *If the coefficients $b_i = b_i(z_*; f),\ i = \overline{1, n}$, of the Thiele continued fraction (10) are nonzero, then the zeros of the canonical denominator $Q_n(z; z_*, f)(z)$ of the Thiele continued fraction for $n > 4$ lie in the disk of radius r_1 centered at the point z_*, where*

$$\max_{1 \le i \le m} |\bar{z}_i - z_*| = r_1 \le \max\left\{2(n-1)(b^*)^{n-2},\ \frac{(n-2)(n-3)}{(n-1)\, b_*^2}\rho\right\},$$

$$b^* = \max_{1 \le i \le n} |b_i|, \qquad b_* = \min_{1 \le i \le n} |b_i|, \qquad \rho = \begin{cases} (b^*/b_*)^{n-4} & \text{for} \quad b^*/b_* \ge 1, \\ (b^*/b_*)^{n-2m} & \text{for} \quad b^*/b_* < 1. \end{cases}$$

Proof. According to (15), the canonical denominator

$$Q_n(z; z_*, f)(z) = q_m(w)$$

is a polynomial of degree m with leading coefficient equal to one. The coefficients $\mathbf{b}_i,\ i = \overline{1, m}$, of the polynomial $q_m(w)$ are not equal to zero. According to Proposition 1, the roots of this polynomial $w_i,\ i = \overline{1, m}$, satisfy inequality (17). If follows from (16) (see [24, 25]) that the coefficient $\mathbf{b}_{m-k}$ is the sum of $\binom{n-k}{k}$ products of $n - 2k$ factors $A_{i_1} A_{i_2} \ldots A_{i_{n-2k}}$.

We have

$$|c_1| \le (n-1)(b^*)^{n-2}, \qquad \left|\frac{c_k}{c_{k-1}}\right| \le \frac{(n+2-2k)(n+1-2k)}{k(n+1-k)\, b_*^2}\left(\frac{b^*}{b_*}\right)^{n-2k}, \quad k = \overline{2, m}.$$

By using (17), we get

$$r_1 \le \max\left\{2|c_1|,\ 2\left|\frac{c_2}{c_1}\right|,\ 2\left|\frac{c_3}{c_2}\right|, \dots,\ 2\left|\frac{c_{m-1}}{c_{m-2}}\right|,\ \left|\frac{c_m}{c_{m-1}}\right|\right\}$$

$$\le \max\left\{2(n-1)(b^*)^{n-2},\ \frac{2(n-2)(n-3)}{2(n-1)\,b_*^2}\left(\frac{b^*}{b_*}\right)^{n-4},\ \frac{2(n-4)(n-5)}{3(n-2)\,b_*^2}\left(\frac{b^*}{b_*}\right)^{n-6}, \dots,\right.$$

$$\left.\frac{2(n+4-2m)(n+3-2m)}{(m-1)(n+2-m)\,b_*^2}\left(\frac{b^*}{b_*}\right)^{n+2-2m},\ \frac{(n+2-2m)(n+1-2m)}{m(n+1-m)\,b_*^2}\left(\frac{b^*}{b_*}\right)^{n-2m}\right\}.$$

Let n be fixed. We consider an auxiliary function

$$g(x) = \frac{(n+1-2x)(n+2-2x)}{x(n+1-x)}, \quad x \in \mathcal{R} = [2; m].$$

The derivative of the function g has the form

$$g'(x) = -\frac{2x^2 - (2n^2+6n+4)x + n^3 + 4n^2 + 5n + 2}{x^2(n+1-x)^2}.$$

On the segment $\mathcal{R}$, the denominator takes only positive values. The numerator is equal to zero if

$$x_1 = \frac{(n+1)(n+2-\sqrt{n^2+2n})}{2}, \qquad x_2 = \frac{(n+1)(n+2+\sqrt{n^2+2n})}{2}.$$

Since $(n^2+2n) > n^2$, we have $x_1 < n+1$. Thus, $g'(x) < 0$ for $x \in \mathcal{R}$, and the function $g(x)$ monotonically decreases on $\mathcal{R}$ and takes the maximum value for $x = 2$.

By using (17), we get

$$r_1 \le \max\left\{2|c_1|,\ 2\left|\frac{c_2}{c_1}\right|, \dots,\ 2\left|\frac{c_{m-1}}{c_{m-2}}\right|,\ \left|\frac{c_m}{c_{m-1}}\right|\right\}$$

$$\le \max\left\{2(n-1)\,(b^*)^{n-2},\ \frac{(n-2)(n-3)}{(n-1)\,b_*^2}\,\rho\right\}.$$

Remark 2. The zeros of the canonical denominator $Q_n(z; z_*, f)$ for $n = 2, 3, 4$ can be found directly.

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