DOI 10.1007/s10958-020-04773-6
Journal of Mathematical Sciences, *Vol. 246, No. 5, May, 2020*

Application of a continuant to the estimation of a remainder term of Thiele's interpolation continued fraction

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Abstract. New properties of a continuant has been proved. Using the relationship between the continuant and the continued fraction, an estimate of the remainder term of Thiele's interpolation continued fraction is obtained.

Keywords. Continued fraction, continuant, interpolation of functions, remainder term.

1. Introduction

The function defined on a compact set $\mathcal{R} \subset \mathbb{R}$ and set by values at points of the set $X \subset \mathcal{R}$ can be approximated by an interpolation polynomial [1,2], spline [3], Padé approximant [4], and interpolation continued fraction [5–8].

In work [9], the formula for the remainder term of Thiele's interpolation continued fraction [10] was generalized to continued fractions whose elements are polynomials. Work [11] gave the estimate of the remainder term of Thiele's interpolation continued fraction in the case where the function $f \in \mathrm{C}^{(n+1)}(\mathcal{R})$.

In the present work, while studying the problem of interpolation of functions of a real variable by of Thiele's interpolation continued fraction, we will use a continuant and prove its properties. We will get the formula for the remainder term of Thiele's interpolation continued fraction and substantiate its estimate. By numerical examples, we will illustrate advantages of the new estimate of a remainder term over that in work [11].

2. Interpolation of functions by continued fractions

Let $b_0, a_k \neq 0, b_k, k \in \mathbb{N}$, be real numbers or functions of a variable x. The infinite continued fraction

$$D = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_k}{b_k + \ddots}}}$$

can be written as follows:

$$b_0 + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots + \frac{a_k}{b_k +} \cdots. \tag{2.1}$$

Translated from Ukrains'kiĭ Matematychnyĭ Visnyk, Vol. 16, No. 4, pp. 588–603 October–December, 2019.
Original article submitted July 25, 2019

Analogously, the nth convergent of continued fraction, the nth approximation of the continued fraction (2.1), can be shortly written as

$$D_n = \frac{P_n}{Q_n} = b_0 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +}\,\cdots +\,\frac{a_n}{b_n}. \tag{2.2}$$

The values of the canonical numerator P_n and denominator Q_n of the continued fraction (2.2) are determined in terms of elements of the continued fraction $b_0, a_k, b_k, k \in \mathbb{N}$, by means of Wallis' formulas [12].

On the compact $\mathcal{R} \subset \mathbb{R}$, let us choose a set of nodes

$$X = \{x_i : x_i \in \mathcal{R}, x_i \neq x_j, i \neq j, i, j = \overline{0, n}\}. \tag{2.3}$$

Let the function f be defined at the nodes of the set X. The function f is interpolated by Thiele's interpolation continued fraction (T–ICF) [5, 13] of the form

$$D_n(x) = \frac{P_n(x)}{Q_n(x)} = b_0 + \mathop{\mathrm{K}}_{i=1}^{n} \frac{x - x_{i-1}}{b_i}, \qquad b_i \in \mathbb{R},\ i = \overline{0, n}. \tag{2.4}$$

T–ICF should satisfy the interpolation conditions

$$D_n(x_i) = y_i,\ x_i \in X,\ y_i = f(x_i),\ i = \overline{0, n}. \tag{2.5}$$

The coefficients of T–ICF (2.4) $b_i, i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, are determined from the interpolation conditions (2.5), or in terms of reciprocal divided differences, or in terms of reciprocal differences [5, 10, 13], or from a recurrence relation in the form of a continued fraction [8]

$$b_0 = y_0, \quad b_1 = \frac{x_1 - x_0}{y_1 - b_0}, \quad b_k = \frac{x_k - x_{k-1}}{-b_{k-1}} + \frac{x_k - x_{k-2}}{-b_{k-2}} + \\ + \cdots + \frac{x_k - x_1}{-b_1} + \frac{x_k - x_0}{y_k - b_0}, \qquad k \in \mathbb{N}_2 = \mathbb{N} \backslash \{1\}.$$

It is known [10] that if the function $f \in \mathbf{C}^{(n+1)}(\mathcal{R})$, then remaider term of T–ICF $R_n(x) = f(x) - D_n(x)$ is given by the formula

$$R_n(x) = \frac{\prod\limits_{i=0}^{n}(x - x_i)}{(n+1)!\, Q_n(x)} \frac{d^{n+1}}{dx^{n+1}} \Big(f(x) \cdot Q_n(x)\Big)\Big|_{x=\xi}, \quad \xi \in \mathcal{R}. \tag{2.6}$$

Theorem 2.1 ([8, 11])**.** *Let $f \in \mathbf{C}^{(n+1)}(\mathcal{R})$. Let T–ICF* (2.4) *be constructed by values of the function at points of set* (2.3)*. Then*

$$\big|R_n(x)\big| \leq E_1 = \frac{f_{max}(b_{max})^n \prod\limits_{k=0}^{n} |x - x_k|}{(n+1)! |Q_n(x)|} \Big(\kappa_{n+1}(\rho) + \sum_{m=1}^{r} \tbinom{n+1}{m} \frac{m!}{b_{min}^{2m}} \times \\ \times \sum_{k=0}^{r-m} \tbinom{n+k}{m} \tbinom{n-m-k}{m+k} \rho^k\Big),$$

where $r = \left[\frac{n}{2}\right]$, $\alpha = \mathbf{diam}\,\mathcal{R}$, $b_{min} = \min\limits_{1\le i\le n} |b_i|$, $b_{max} = \max\limits_{1\le i\le n} |b_i|$, $\rho = \dfrac{\alpha}{b_{min}^2}$,

$$f_{max} = \max_{0\le m\le r}\max_{x\in\mathcal{R}} |f^{(n+1-m)}(x)|, \kappa_n(\rho) = \frac{(1+\sqrt{1+4\rho})^n - (1-\sqrt{1+4\rho})^n}{2^n\sqrt{1+4\rho}}.$$

The main result of the present work is the following proposition.

Theorem 2.2. *Let* $f \in \mathbf{C}^{(n+1)}(\mathcal{R})$. *Let T–ICF* (2.4) *be constructed by values of the function* f *at points of set* (2.3). *Then*

$$\big|R_n(x)\big| \le E_2 = \frac{f_{max}\prod\limits_{k=0}^{n}|x-x_k|}{(n+1)!\,|\mathbf{T}_n^{\langle 1\rangle}(x)|}\Bigg(b_{max}^n\,\kappa_{n+1}(\rho) + \sum_{k=1}^{r}\binom{n+1}{k} b_{max}^{n-2k}\times$$

$$\times\sum_{i_1=1}^{n+1-2k}\kappa_{i_1}(\rho)\sum_{i_2=i_1+2}^{n+3-2k}\kappa_{i_2-i_1-1}(\rho)\cdots\sum_{i_{k-1}=i_{k-2}+2}^{n-3}\kappa_{i_{k-1}-i_{k-2}-1}(\rho)\times$$

$$\times\sum_{i_k=i_{k-1}+2}^{n-1}\kappa_{i_k-i_{k-1}-1}(\rho)\kappa_{n-i_k}(\rho)\Bigg), \tag{2.7}$$

where $\mathbf{T}_n^{\langle 1\rangle}(x)$ *is a continuant which is defined in* (4.2).

3. Continuant and its properties

Let $b_0, a_i, b_i, i \in \mathbb{N}$, be real numbers or functions. Consider the determinant

$$\mathcal{H}_n^{\langle i\rangle} = \begin{vmatrix} b_i & a_{i+1} & 0 & 0 & \dots & 0 & 0 \\ -1 & b_{i+1} & a_{i+2} & 0 & \dots & 0 & 0 \\ 0 & -1 & b_{i+2} & a_{i+3} & \dots & 0 & 0 \\ 0 & 0 & -1 & b_{i+3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & b_{n-1} & a_n \\ 0 & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix}, \ i = \overline{0,n},\ n \in \mathbb{N}, \tag{3.1}$$

which is called a continuant [14] and is briefly written as

$$\mathcal{H}_n^{\langle i\rangle} = \mathcal{K}\begin{pmatrix} a_{i+1}, a_{i+2}, \dots, a_{n-1}, a_n \\ b_i, b_{i+1}, b_{i+2}, \dots, b_{n-1}, b_n \end{pmatrix}.$$

Continuant (3.1) as a partial case of the Hessenberg determinant satisfies the three-term recurrence relation [15]

$$\mathcal{H}_m^{\langle i\rangle} = b_m\mathcal{H}_{m-1}^{\langle i\rangle} + a_m\mathcal{H}_{m-2}^{\langle i\rangle},\ m = \overline{i+1,n},\ H_i^{\langle i\rangle} = b_i,\ H_{i-1}^{\langle i\rangle} = 1. \tag{3.2}$$

Theorem 3.1. *If an element* $a_k, i < k \le n$, *of the continuant* $\mathcal{H}_n^{\langle i\rangle}$ *is equal to zero, and the remaining elements are nonzero, then*

$$\mathcal{H}_n^{\langle i\rangle} = \mathcal{H}_n^{\langle k\rangle}\cdot\mathcal{H}_{k-1}^{\langle i\rangle}. \tag{3.3}$$

Proof. Since $a_k = 0$, the recurrence relation (3.2) yields

$$\mathcal{H}_k^{\langle i\rangle} = b_k \mathcal{H}_{k-1}^{\langle i\rangle} = \mathcal{H}_k^{\langle k\rangle}\mathcal{H}_{k-1}^{\langle i\rangle},$$

$$\mathcal{H}_{k+1}^{\langle i\rangle} = b_{k+1}\mathcal{H}_k^{\langle i\rangle} + a_{k+1}\mathcal{H}_{k-1}^{\langle i\rangle} = (b_k b_{k+1} + a_{k+1})\mathcal{H}_{k-1}^{\langle i\rangle} = \mathcal{H}_{k+1}^{\langle k\rangle}\mathcal{H}_{k-1}^{\langle i\rangle}.$$

Hence, formula (3.3) is true for $n = k$ and $n = k+1$. Assume that relations (3.3) hold for $n = m$. From (3.2), we get

$$\mathcal{H}_{m+1}^{\langle i\rangle} = b_{m+1}\mathcal{H}_m^{\langle i\rangle} + a_{m+1}\mathcal{H}_{m-1}^{\langle i\rangle} =$$

$$= b_{m+1}\mathcal{H}_m^{\langle k\rangle}\mathcal{H}_{k-1}^{\langle i\rangle} + a_{m+1}\mathcal{H}_{m-1}^{\langle k\rangle}\mathcal{H}_{k-1}^{\langle i\rangle} = \mathcal{H}_{m+1}^{\langle k\rangle}\mathcal{H}_{k-1}^{\langle i\rangle}.$$

Hence, formula (3.3) holds for any n. □

We now prove the following proposition.

Theorem 3.2. *If elements of the continuant $a_s \neq 0, s = \overline{i,n}$, $b_s \neq 0$, $s = \overline{i, k+i-2}$, $b_t \neq 0, t = \overline{k+i, n}$, and $b_{k+i-1} = 0, i \leq n, k = \overline{1, n-i}$, then the continuant*

$$\mathcal{A}_n^{\langle i,k\rangle} = \begin{vmatrix} b_i & a_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & b_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{k+i-2} & a_{k+i-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & a_{k+i} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & b_{k+i} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & b_{n-1} & a_n \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix} =$$

$$= a_{k+i}\mathcal{H}_{k+i-2}^{\langle i\rangle}\mathcal{H}_n^{\langle k+i+1\rangle}, \quad \textit{where } \mathcal{H}_n^{\langle n+1\rangle} = \mathcal{H}_0^{\langle i\rangle} = 1,\ k = \overline{1, n-i}. \tag{3.4}$$

Proof. Let $k = 1$. Then

$$\mathcal{A}_n^{\langle i,1\rangle} = \begin{vmatrix} 0 & a_{i+1} & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & b_{i+1} & a_{i+2} & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & b_{i+2} & a_{i+3} & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & b_{i+3} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & b_{n-2} & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & b_{n-1} & a_n \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & b_n \end{vmatrix}.$$

If we expand the determinant successively in the 1st row and 1st column, we get $\mathcal{A}_n^{\langle i,1\rangle} = a_{i+1}\mathcal{H}_{i-1}^{\langle i\rangle}\mathcal{H}_n^{\langle i+2\rangle}$. For $k = 2$, we expand the determinant successively in the 2nd row and the 2nd column. We get $\mathcal{A}_n^{\langle i,2\rangle} = a_{i+2}\mathcal{H}_i^{\langle i\rangle}\mathcal{H}_n^{\langle i+3\rangle}$. In the general case $k = m$, we expand the determinant successively in the mth row and mth column:

$$\mathcal{A}_n^{\langle i,m\rangle} = \begin{vmatrix} b_i & a_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & b_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{m+i-2} & a_{m+i-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & a_{m+i} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & b_{m+i} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & b_{n-1} & a_n \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix} =$$

$$= -a_{m+i} \begin{vmatrix} b_i & a_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & b_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{m+i-2} & a_{m+i-1} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & a_{m+i+1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & b_{m+i+1} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & b_{n-1} & a_n \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix} =$$

$$= -a_{m+i}a_{m+i-1} \begin{vmatrix} b_i & a_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 \\ -1 & b_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{m+i-3} & a_{m+i-2} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & a_{m+i+1} & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & b_{m+i+1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & b_n \end{vmatrix} +$$

$$+ a_{m+i} \begin{vmatrix} b_i & a_{i+1} & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & b_{i+1} & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{m+i-3} & a_{m+i-2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & -1 & b_{m+i-2} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & b_{m+i+1} & a_{m+i+2} & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 & b_{m+i+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & b_n \end{vmatrix}.$$

Taking $(m-2)$ first columns in the determinants and using the Laplace theorem [16], we obtain $\mathcal{A}_n^{\langle i,m\rangle} = a_{m+i}\mathcal{H}_{m+i-2}^{\langle i\rangle}\mathcal{H}_n^{\langle m+i+1\rangle}$. Formula (3.4) is proved. $\square$

A continuant is invariant relative to the inversion of the order of elements [14], i.e.,

$$\mathcal{K}\begin{pmatrix} & a_{i+1}, a_{i+2}, \dots, a_{n-1}, a_n \\ b_i, b_{i+1}, & b_{i+2}, \dots, b_{n-1}, b_n \end{pmatrix} = \mathcal{K}\begin{pmatrix} & a_n, \ a_{n-1}, \dots, \ a_{i+2}, a_{i+1} \\ b_n, b_{n-1}, & b_{n-2}, \dots, \ b_{i+1}, \ b_i \end{pmatrix}.$$

Theorem 3.3. *If $\mathcal{H}_n^{\langle i\rangle} \neq 0$, where $i = 0, 1$, and if $w = \mathcal{H}_n^{\langle 0\rangle}/\mathcal{H}_n^{\langle 1\rangle}$, then*

$$b_n = \frac{a_n \mathcal{K}\begin{pmatrix} a_{n-2}, & a_{n-3}, & \dots, & a_3, & a_2, & a_1 \\ -b_{n-2}, & -b_{n-3}, & -b_{n-4}, & \dots, & -b_2, & -b_1, w-b_0 \end{pmatrix}}{\mathcal{K}\begin{pmatrix} a_{n-1}, & a_{n-2}, & \dots, & a_3, & a_2, & a_1 \\ -b_{n-1}, & -b_{n-2}, & -b_{n-3}, & \dots, & -b_2, & -b_1, w-b_0 \end{pmatrix}}. \tag{3.5}$$

Proof. Since $w = \mathcal{H}_n^{\langle 0\rangle}/\mathcal{H}_n^{\langle 1\rangle}$ by assumptions of the theorem, we have

$$w\,\mathcal{K}\begin{pmatrix} a_n, & a_{n-1}, & \dots, & a_3, a_2 \\ b_n, b_{n-1}, & b_{n-2}, & \dots, & b_2, b_1 \end{pmatrix} = \mathcal{K}\begin{pmatrix} a_n, & a_{n-1}, & \dots, & a_2, a_1 \\ b_n, b_{n-1}, & b_{n-2}, & \dots, & b_1, b_0 \end{pmatrix}.$$

We now expand the determinants in the 1st column:

$$w\,b_n\,\mathcal{K}\begin{pmatrix} a_{n-1}, & \dots, & a_3, a_2 \\ b_{n-1}, b_{n-2}, & \dots, & b_2, b_1 \end{pmatrix} + w\,a_n\,\mathcal{K}\begin{pmatrix} a_{n-2}, & \dots, & a_3, a_2 \\ b_{n-2}, b_{n-3}, & \dots, & b_2, b_1 \end{pmatrix} =$$

$$= b_n\,\mathcal{K}\begin{pmatrix} a_{n-1}, & \dots, & a_2, a_1 \\ b_{n-1}, b_{n-2}, & \dots, & b_1, b_0 \end{pmatrix} + a_n\,\mathcal{K}\begin{pmatrix} a_{n-2}, & \dots, & a_2, a_1 \\ b_{n-2}, b_{n-3}, & \dots, & b_1, b_0 \end{pmatrix}.$$

From whence, we get

$$b_n = \frac{-\,a_n\,\mathcal{K}\begin{pmatrix} a_{n-2}, & \dots, & a_2, & a_1 \\ b_{n-2}, b_{n-3}, & \dots, & b_1, & b_0 - w \end{pmatrix}}{\mathcal{K}\begin{pmatrix} a_{n-1}, & \dots, & a_2, & a_1 \\ b_{n-1}, b_{n-2}, & \dots, & b_1, & b_0 - w \end{pmatrix}}.$$

Let us take out the factor (-1) from odd rows and even columns of the determinants of the numerator and denominator. We get (3.5). $\square$

4. Interrelation of a continuant and an interpolation continued fraction

It is known [15] that the nth convergent D_n of the continued fraction (2.1) can be presented in the form of a ratio of continuants, i.e.,

$$D_n = \frac{P_n}{Q_n} = \frac{\mathcal{H}_n^{\langle 0\rangle}}{\mathcal{H}_n^{\langle 1\rangle}}. \tag{4.1}$$

Introduce the following continuants:

$$\mathbf{T}_m^{\langle i\rangle}(x) = \mathcal{K}\begin{pmatrix} x - x_i, & x - x_{i+1}, & \dots, & x - x_{m-1} \\ b_i, & b_{i+1}, & b_{i+2}, & \dots, & b_m \end{pmatrix}, \quad i < m. \tag{4.2}$$

From (4.1), we have that T–ICF (2.4) can be written as the ratio of continuants of the form (4.2): $D_n(x) = \mathbf{T}_n^{\langle 0\rangle}(x)/\mathbf{T}_n^{\langle 1\rangle}(x)$. In addition, for arbitrary value of $k = \overline{1, n}$, the following relation holds:

$$D_n(x_k) = \frac{\mathbf{T}_k^{\langle 0\rangle}(x_k)}{\mathbf{T}_k^{\langle 1\rangle}(x_k)}. \tag{4.3}$$

Relation (4.3) follows directly from Theorem 3.1. Since the element a_{k+i} of the continuant $\mathbf{T}_n^{\langle i\rangle}(x_k), i = 0, 1$, is equal to zero, (3.3) yields

$$D_n(x_k) = \frac{\mathbf{T}_k^{\langle 0\rangle}(x_k)\,\mathbf{T}_n^{\langle k+1\rangle}(x_k)}{\mathbf{T}_k^{\langle 1\rangle}(x_k)\,\mathbf{T}_n^{\langle k+1\rangle}(x_k)} = \frac{\mathbf{T}_k^{\langle 0\rangle}(x_k)}{\mathbf{T}_k^{\langle 1\rangle}(x_k)}.$$

Theorem 4.1. *The coefficients of T–ICF* (2.4) *are determined by the formulas*

$$b_0 = y_0, \quad b_1 = \frac{x_1 - x_0}{y_1 - b_0}, \quad b_2 = \frac{\begin{vmatrix} x_2 - x_1 & 0 \\ -1 & y_2 - b_0 \end{vmatrix}}{\begin{vmatrix} -b_1 & x_2 - x_0 \\ -1 & y_2 - b_0 \end{vmatrix}}, \tag{4.4}$$

$$b_k = \frac{(x_k - x_{k-1})\,\mathcal{K}\begin{pmatrix} & x_k - x_{k-3}, & \ldots, & x_k - x_1, & x_k - x_0 \\ -b_{k-2}, & -b_{k-3}, & \ldots, & -b_1, & y_k - b_0 \end{pmatrix}}{\mathcal{K}\begin{pmatrix} & x_k - x_{k-2}, & \ldots, & x_k - x_1, & x_k - x_0 \\ -b_{k-1}, & -b_{k-2}, & \ldots, & -b_1, & y_k - b_0 \end{pmatrix}}, \tag{4.5}$$

if $k = \overline{3, n}$.

Proof. From the interpolation condition, we have $y_k = D_n(x_k)$, $k = \overline{0, n}$. It follows from (4.3) that $y_k = \mathbf{T}_k^{\langle 0 \rangle}(x_k)/\mathbf{T}_k^{\langle 1 \rangle}(x_k), k = \overline{0, n}$. According to Theorem 3.3, the coefficient b_k is given by formula (3.5), where $a_i = x_k - x_{i-1}, i = \overline{1, k}$. From whence, we get (4.4)–(4.5). $\square$

5. Formula for the remainder term of T–ICF

Let us determine the remainder term of T–ICF in terms of a continuant. Consider the determinants $\left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_i$, where $i = \overline{1, n}$. They are formed from the continuant $\mathbf{T}_n^{\langle 1 \rangle}(x)$ by the replacement of elements of the ith row by their derivatives. It is obvious that the ith row $\left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_i$ includes the single nonzero element equal to 1 on the cross with the $(i+1)$th column, $i = \overline{0, n-1}$. Obviously, $\left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_n \equiv 0$, since the last row of the determinant contains only zeros. Let $\left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_{ij}$, $i, j = \overline{1, n}$, be determinants formed from the continuant $\mathbf{T}_n^{\langle 1 \rangle}(x)$ by the replacement of elements of the ith and jth rows by their derivatives. The following identities hold:

$$\begin{aligned} \left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_{ii} \equiv \left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_{i,n} \equiv 0, \quad i = \overline{1, n-2}, \\ \left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_{i,i+1} \equiv 0, \qquad i = \overline{1, n-1}. \end{aligned} \tag{5.1}$$

The first identity (5.1) is obvious, since each determinant contains one row with zero elements. We get the second identity, by expanding the determinants by the Laplace rule in the sum of products of the second-order minors that are contained in the ith and $(i+1)$th rows by their cofactors.

Let $\left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_{i_1 i_2 \ldots i_k}$ be the determinant formed from the continuant $\mathbf{T}_n^{\langle 1 \rangle}(x)$ by the replacement of elements of the rows $i_1, i_2, \ldots, i_k$ by their derivatives.

Theorem 5.1. (A) *The derivative of the kth order, $k = \overline{1, [n/2]}$, of the continuant $\mathbf{T}_n^{\langle 1 \rangle}(x)$ is equal to*

$$\left(\mathbf{T}_n^{\langle 1 \rangle}(x)\right)^{(k)} = k! \sum_{i_1=1}^{n+1-2k} \sum_{i_2=i_1+2}^{n+3-2k} \cdots \sum_{i_k=i_{k-1}+2}^{n-1} \left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_{i_1 i_2 \ldots i_k}. \tag{5.2}$$

(B) *If $k > [n/2]$, then $\left(\mathbf{T}_n^{\langle 1 \rangle}(x)\right)^{(k)} \equiv 0$.*

Proof. **(A)** Formula (5.2) can be proved by induction. Since the identity $\left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_n \equiv 0$ holds, the rule of differentiation of determinants for $k = 1$ yields

$$\left(\mathbf{T}_n^{\langle 1 \rangle}(x)\right)^{(1)} = \sum_{i=1}^{n-1} \left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_i, \quad \left[\mathbf{t}_n^{\langle 1 \rangle}(x)\right]_n \equiv 0.$$

The second derivative of the continuant $\mathbf{T}_n^{\langle 1\rangle}(x)$ is

$$\left(\mathbf{T}_n^{\langle 1\rangle}(x)\right)^{(2)} = \sum_{i_2=1}^{n}\sum_{i_1=1}^{n-1}\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_1 i_2}.$$

Accounting for the symmetry of the determinants $\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{ij} = \left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{ji}$ and basing on relation (5.1), we have

$$\left(\mathbf{T}_n^{\langle 1\rangle}(x)\right)^{(2)} = 2\cdot\sum_{i_1=1}^{n-3}\sum_{i_2=i_1+2}^{n-1}\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_1 i_2}.$$

Assume that (5.2) holds for $k=m-1$, where $m-1<[n/2]$.

We now find the derivative of the mth order of the continuant $\mathbf{T}_n^{\langle 1\rangle}(x)$. We have

$$\left(\left(\mathbf{T}_n^{\langle 1\rangle}(x)\right)^{(m-1)}\right)' = (m-1)!\sum_{i_m=1}^{n}\sum_{i_1=1}^{n+3-2m}\cdots\sum_{i_{m-1}=i_{m-2}+2}^{n-1}\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_1\ldots i_m}.$$

In view of (5.1) and the relations

$$\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_1 i_2\ldots i_{m-1} i_m} = \left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_2 i_1\ldots i_{m-1} i_m} = \ldots = \left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_m i_{m-1}\ldots i_2 i_1},$$

we get

$$\left(\mathbf{T}_n^{\langle 1\rangle}(x)\right)^{(m)} = m!\sum_{i_1=1}^{n+1-2m}\sum_{i_2=i_1+2}^{n+3-2m}\cdots\sum_{i_m=i_{m-1}+2}^{n-1}\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_1 i_2\ldots i_m}.$$

Hence, formula (5.2) holds for any $1\le m\le [n/2]$.

(B) According to item **(A)**, the derivative of the order $l=[n/2]$ of the continuant $\mathbf{T}_n^{\langle 1\rangle}(x)$ is given by formula (5.2).

We now find the derivative of the $(l+1)$th order:

$$\left(\mathbf{T}_n^{\langle 1\rangle}(x)\right)^{(l+1)} = l!\sum_{k=1}^{n}\sum_{i_1=1}^{n+1-2l}\sum_{i_2=i_1+2}^{n+3-2l}\cdots\sum_{i_l=i_{l-1}+2}^{n-1}\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_1 i_2\ldots i_m k}.$$

According to (5.1), all determinants on the right-hand side are zero. □

Theorem 5.2. *Let the function $f\in\mathbf{C}^{(n+1)}(\mathcal{R})$ be interpolated by its values at points of set (2.3) for T–ICF (2.4). Then there exists $\xi\in\mathbf{Int}\,\mathcal{R}$ such that the remainder term $R_n(x)=f(x)-D_n(x)$ is determined by the formula*

$$R_n(x) = \frac{\prod_{i=0}^{n}(x-x_i)}{(n+1)!\,\mathbf{T}_n^{\langle 1\rangle}(x)}\Bigg(f^{(n+1)}(x)\mathbf{T}_n^{\langle 1\rangle}(x) + \sum_{k=1}^{r}\binom{n+1}{k} f^{(n+1-k)}(x)\times$$
$$\times\sum_{i_1=1}^{n+1-2k}\sum_{i_2=i_1+2}^{n+3-2k}\cdots\sum_{i_k=i_{k-1}+2}^{n-1}\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_1 i_2\ldots i_k}\Bigg)\Bigg|_{x=\xi},\qquad r=[n/2]. \tag{5.3}$$

Proof. We have $R_n(x) = f(x) - \mathbf{T}_n^{\langle 0\rangle}(x)/\mathbf{T}_n^{\langle 1\rangle}(x)$. Consider the auxiliary function

$$F(x) = f(x)\,\mathbf{T}_n^{\langle 1\rangle}(x) - \mathbf{T}_n^{\langle 0\rangle}(x) - \lambda(x-x_0)(x-x_1)\dots(x-x_n). \tag{5.4}$$

According to the interpolation conditions (2.5), the function F is equal to zero at the $(n+1)$th point $x_i \in \mathcal{R}$, $i = \overline{0,n}$. If the parameter λ is chosen in the form

$$\lambda = \frac{f(x_*)\,\mathbf{T}_n^{\langle 1\rangle}(x_*) - \mathbf{T}_n^{\langle 0\rangle}(x_*)}{(x_*-x_0)(x_*-x_1)\dots(x_*-x_n)}, \qquad \text{where} \quad x_* \in \mathcal{R}\backslash X,$$

then the function F is equal to zero at $(n+2)$ points of the extended set $\tilde{X} = X \cup \{x_*\} \subset \mathcal{R}$. According to Rolle's generalized theorem [1], there exists a point $\xi \in \mathbf{Int}\,\mathcal{R}$ such that $F^{(n+1)}(\xi) = 0$ or

$$\frac{d^{n+1}}{dx^{n+1}}\Big(f(x)\mathbf{T}_n^{\langle 1\rangle}(x)\Big)\Big|_{x=\xi} - \frac{d^{n+1}}{dx^{n+1}}\Big(\mathbf{T}_n^{\langle 0\rangle}(x)\Big)\Big|_{x=\xi} - (n+1)!\lambda = 0.$$

Theorem 5.1 implies that $\big(\mathbf{T}_n^{\langle 0\rangle}(x)\big)^{(n+1)} \equiv 0$. Using the formula for a derivative of the $(n+1)$th order of the product of two functions, we get

$$\frac{d^{n+1}}{dx^{n+1}}\Big(f(x)\cdot\mathbf{T}_n^{\langle 1\rangle}(x)\Big) = \sum_{k=0}^{r}\tbinom{n+1}{k} f^{(n+1-k)}(x)\big(\mathbf{T}_n^{\langle 1\rangle}(x)\big)^{(k)}.$$

Then

$$\lambda = \frac{1}{(n+1)!}\sum_{k=0}^{r}\Big(\tbinom{n+1}{k}\, f^{(n+1-k)}(x)\times$$
$$\times \sum_{i_1=1}^{n+1-2k}\ \sum_{i_2=i_1+2}^{n+3-2k}\cdots\sum_{i_k=i_{k-1}+2}^{n-1}\big[\mathbf{t}_n^{\langle 1\rangle}(x)\big]_{i_1i_2\dots i_k}\Big)\Big|_{x=\xi}.$$

Since x_* is arbitrary point from $\mathcal{R}$, we divide (5.4) by $\mathbf{T}_n^{\langle 1\rangle}(x)$ and arrive at (5.3). □

Remark 5.1. The formula for a remainder term (5.3) is given in terms of the continuant $\mathbf{T}_n^{\langle 1\rangle}(x)$ and its derivatives and differs from formula (2.6) for a remainder term. This gives possibilities to substantiate the more exact estomates of the remainder term of T–ICF.

6. Proof of Theorem 2.2

Here, we will prove the main result of the present work, namely, Theorem 2.2.

Proof. We present formula (5.3) from Theorem 5.2 in another form. To this end, we write the determinants $\big[\mathbf{t}_n^{\langle 1\rangle}(x)\big]_{i_1i_2\dots i_k}$ in terms of the continuant $\mathbf{T}_s^{\langle l\rangle}(x)$ for some values of l and s. We note that the ith row of the determinant $\big[\mathbf{t}_n^{\langle 1\rangle}(x)\big]_i$, for $i = \overline{1,n-1}$, contains zeros except for one element which is placed on the intersection with the $(i+1)$th column and equals 1. Then, according to Theorem 3.2, we have $\big[\mathbf{t}_n^{\langle 1\rangle}(x)\big]_i = \mathbf{T}_{i-1}^{\langle 1\rangle}(x)\mathbf{T}_n^{\langle i+2\rangle}(x)$. In the determinant $\big[\mathbf{t}_n^{\langle 1\rangle}(x)\big]_{i_1i_2}$, the single nonzero element of the i_1th row is equal to 1 and is located in the (i_1+1)th position. Therefore, according to the same theorem, we have $\big[\mathbf{t}_n^{\langle 1\rangle}(x)\big]_{i_1i_2} = \mathbf{T}_{i_1-1}^{\langle 1\rangle}(x)\,\mathbf{A}_n^{\langle i_1+2,i_2\rangle}(x)$, where $\mathbf{A}_n^{\langle i_1+2,i_2\rangle}(x)$ follows from $\mathcal{A}_n^{\langle i_1+2,i_2\rangle}(x)$ by the replacement of a_{i+1} by $x - x_i$. The single nonzero element of the

$(i_2 - i_1 - 1)$th row of the determinant $\mathbf{A}_n^{\langle i_1+2, i_2\rangle}(x)$ is equal to 1 and occupies the $(i_2 - i_1)$th position. Then $\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_1 i_2} = \mathbf{T}_{i_1-1}^{\langle 1\rangle}(x)\mathbf{T}_{i_2-1}^{\langle i_1+2\rangle}(x)\mathbf{T}_n^{\langle i_2+2\rangle}(x)$. Analogously, we can show that

$$\left[\mathbf{t}_n^{\langle 1\rangle}(x)\right]_{i_1 i_2 \dots i_s} = \mathbf{T}_{i_1-1}^{\langle 1\rangle}(x)\mathbf{T}_{i_2-1}^{(i_1+2)}(x)\cdots\mathbf{T}_{i_s-1}^{(i_{s-1}+2)}(x)\mathbf{T}_n^{(i_s+2)}(x), \tag{6.1}$$

where $s = \overline{1, r}$, $\mathbf{T}_{s-1}^{(s)}(x) = 1$.

With regard for (6.1), formula (5.3) takes the form

$$R_n(x) = \frac{\prod\limits_{k=0}^{n}(x - x_k)}{(n+1)!\,\mathbf{T}_n^{\langle 1\rangle}(x)}\Bigg(f^{(n+1)}(\xi)\mathbf{T}_n^{\langle 1\rangle}(\xi) + f^{(n)}(\xi)\tbinom{n+1}{1}\times$$

$$\times\sum_{i=1}^{n-1}\mathbf{T}_{i-1}^{\langle 1\rangle}(\xi)\mathbf{T}_n^{\langle i+2\rangle}(\xi) + \tbinom{n+1}{2}f^{(n-1)}(\xi)\sum_{i_1=1}^{n-3}\sum_{i_2=i_1+2}^{n-1}\mathbf{T}_{i_1-1}^{\langle 1\rangle}(\xi)\mathbf{T}_{i_2-1}^{\langle i_1+2\rangle}(\xi)\times$$

$$\times\mathbf{T}_n^{\langle i_2+2\rangle}(\xi) + \cdots + \tbinom{n+1}{r}f^{(n+1-r)}(\xi)\sum_{i_1=1}^{n+1-2r}\sum_{i_2=i_1+2}^{n+3-2r}\cdots\sum_{i_r=i_{r-1}+2}^{n-1}\mathbf{T}_{i_1-1}^{\langle 1\rangle}(\xi)\times$$

$$\times\mathbf{T}_{i_2-1}^{\langle i_1+2\rangle}(\xi)\cdots\mathbf{T}_{i_r-1}^{\langle i_{r-1}+2\rangle}(\xi)\mathbf{T}_n^{\langle i_r+2\rangle}(\xi)\Bigg). \tag{6.2}$$

We now substantiate estimate (2.7). Let us evaluate (6.2) in modulus. We have

$$\left|R_n(x)\right| \le \frac{\prod\limits_{k=0}^{n}|x - x_k|}{(n+1)!\left|\mathbf{T}_n^{\langle 1\rangle}(x)\right|}\Big(\left|f^{(n+1)}(\xi)\right|\left|\mathbf{T}_n^{\langle 1\rangle}(\xi)\right| + \tbinom{n+1}{1}\left|f^{(n)}(\xi)\right|\times$$

$$\times\sum_{i=1}^{n-1}\left|\mathbf{T}_{i-1}^{\langle 1\rangle}(\xi)\right|\left|\mathbf{T}_n^{\langle i+2\rangle}(\xi)\right| + \tbinom{n+1}{2}\left|f^{(n-1)}(\xi)\right|\sum_{i_1=1}^{n-3}\sum_{i_2=i_1+2}^{n-1}\left|\mathbf{T}_{i_1-1}^{\langle 1\rangle}(\xi)\right|\times$$

$$\times\left|\mathbf{T}_{i_2-1}^{\langle i_1+2\rangle}(\xi)\right|\left|\mathbf{T}_n^{\langle i_2+2\rangle}(\xi)\right| + \cdots + \tbinom{n+1}{r}\left|f^{(n+1-r)}(\xi)\right|\sum_{i_1=1}^{n+1-2r}\sum_{i_2=i_1+2}^{n+3-2r}\cdots$$

$$\cdots\sum_{i_r=i_{r-1}+2}^{n-1}\left|\mathbf{T}_{i_1-1}^{\langle 1\rangle}(\xi)\right|\left|\mathbf{T}_{i_2-1}^{\langle i_1+2\rangle}(\xi)\right|\cdots\left|\mathbf{T}_{i_r-1}^{\langle i_{r-1}+2\rangle}(\xi)\right|\left|\mathbf{T}_n^{\langle i_r+2\rangle}(\xi)\right|\Big).$$

In [8], it was proved that $\left|\mathbf{T}_t^{\langle s\rangle}(x)\right| < (b_{max})^{t-s+1}\kappa_{t-s+2}(\rho)$, $s \le t$. Then

$$\left|R_n(x)\right| \le \frac{f_{max}\prod\limits_{k=0}^{n}|x - x_k|}{(n+1)!\left|\mathbf{T}_n^{\langle 1\rangle}(x)\right|}\Big((b_{max})^n\kappa_{n+1}(\rho) + \tbinom{n+1}{1}(b_{max})^{n-2}\times$$

$$\times\sum_{i=1}^{n-1}\kappa_i(\rho)\kappa_{n-i}(\rho) + \tbinom{n+1}{2}(b_{max})^{n-4}\sum_{i_1=1}^{n-3}\kappa_{i_1}(\rho)\sum_{i_2=i_1+2}^{n-1}\kappa_{i_2-i_1-1}(\rho)\times$$

$$\times\kappa_{n-i_2}(\rho)+\binom{n+1}{3}(b_{max})^{n-6}\sum_{i_1=1}^{n-5}\kappa_{i_1}(\rho)\sum_{i_2=i_1+2}^{n-3}\kappa_{i_2-i_1-1}(\rho)\times$$

$$\times\sum_{i_3=i_2+2}^{n-1}\kappa_{i_3-i_2-1}(\rho)\kappa_{n-i_3}(\rho)+\cdots+\binom{n+1}{r}(b_{max})^{n-2r}\sum_{i_1=1}^{n+1-2r}\kappa_{i_1}(\rho)\times$$

$$\times\sum_{i_2=i_1+2}^{n+3-2r}\kappa_{i_2-i_1-1}(\rho)\cdots\sum_{i_r=i_{r-1}+2}^{n-1}\kappa_{i_r-i_{r-1}-1}(\rho)\kappa_{n-i_r}(\rho)\Big).$$

Estimate (2.7) is proved. □

7. Numerical examples

To illustrate the efficiency of estimate E_2 of the remainder term of T–ICF (5.3) and to compare it with estimate E_1 from Theorem 2.1 for the remainder term of T–ICF in the form (2.6), we now consider some numerical examples.

As interpolation nodes, we choose the roots of a Chebyshev polynomial of the fist kind which are belonging to $\mathcal{R}$. It is well known [1] that, for such choice of nodes, $\prod_{k=0}^{n}|x-x_k|\le\frac{\alpha^{n+1}}{2^{2n+1}}$, $\alpha=$ **diam** $\mathcal{R}$. In addition, we assume that the coefficients $b_i, i=\overline{1,n}$, of T–ICF (2.4) satisfy the Śleszyński–Pringsheim condition, i.e., $|b_i|\ge\alpha+1, i=\overline{1,n}$.

In [8], it was substantiated that if the elements $a_i(x), b_i(x), i=\overline{1,n}$, of a continued fraction $\mathbf{K}_{i=1}^{n}(a_i(x)/b_i(x))$ satisfy the conditions $|a_i(x)|\le\alpha$, $|b_i(x)|\ge\alpha+1, i=\overline{1,n}, x\in\mathcal{R}$, then $Q_n(x)$ and $\mathbf{T}_n^{\langle 1\rangle}(x)$ satisfy the inequality

$$|Q_n(x)|\ge\Lambda_n,\ |\mathbf{T}_n^{\langle 1\rangle}(x)|\ge\Lambda_n,\ \text{where } \Lambda_n=\begin{cases}\frac{\alpha^{n+1}-1}{\alpha-1}, & \alpha\ne 1,\\ n+1, & \alpha=1.\end{cases}$$

If the roots of a Chebyshev polynomial of the fist kind are chosen as nodes, the coefficients of T–ICF satisfy the condition $|b_i|\ge\alpha+1, i=\overline{1,n}$, and estimate E_1 takes the form

$$E_1=\frac{f_{max}b_{max}^{n}\alpha^{n+1}}{2^{2n+1}\Lambda_n(n+1)!}\Big(\kappa_{n+1}(\rho)+\sum_{m=1}^{r}\binom{n+1}{m}\frac{m!}{b_{min}^{2m}}\sum_{k=0}^{r-m}\binom{n+k}{m}\binom{n-m-k}{m+k}\rho^k\Big),$$

where $\kappa_n(\rho), f_{max}, b_{min}, b_{max}, \rho, r, \alpha$ are defined in the condition of Theorem 2.1.

Analogously, estimate E_2 will be overwritten

$$E_2=\frac{f_{max}\cdot\alpha^{n+1}}{2^{2n+1}\Lambda_n(n+1)!}\Bigg(b_{max}^{n}\kappa_{n+1}(\rho)+\sum_{k=1}^{r}\binom{n+1}{k}b_{max}^{n-2k}\times$$

$$\times\sum_{i_1=1}^{n+1-2k}\kappa_{i_1}(\rho)\sum_{i_2=i_1+2}^{n+3-2k}\kappa_{i_2-i_1-1}(\rho)\cdots\sum_{i_{k-1}=i_{k-2}+2}^{n-3}\kappa_{i_{k-1}-i_{k-2}-1}(\rho)\times$$

$$\times\sum_{i_k=i_{k-1}+2}^{n-1}\kappa_{i_k-i_{k-1}-1}(\rho)\kappa_{n-i_k}(\rho)\Bigg).$$

Table 9.1: Function 2^x, segment $\mathcal{R} = [-0.45, 0.0]$

n	b_{min}	b_{max}	E_1	E_2
4	1.495319	4.744028	$0.37622386 \cdot 10^{-03}$	$0.68695834 \cdot 10^{-04}$
5	1.479851	8.216546	$0.94634967 \cdot 10^{-03}$	$0.65432811 \cdot 10^{-04}$
6	1.470265	8.027247	$0.33122929 \cdot 10^{-03}$	$0.97885121 \cdot 10^{-05}$
7	1.463939	11.569403	$0.10333131 \cdot 10^{-02}$	$0.10578746 \cdot 10^{-04}$
8	1.459553	11.356674	$0.43434560 \cdot 10^{-03}$	$0.16740396 \cdot 10^{-05}$
9	1.456393	14.932188	$0.15610864 \cdot 10^{-02}$	$0.19200421 \cdot 10^{-05}$
10	1.454042	14.705507	$0.75069601 \cdot 10^{-03}$	$0.31442508 \cdot 10^{-06}$
11	1.452246	18.299149	$0.29980084 \cdot 10^{-02}$	$0.37312679 \cdot 10^{-06}$
12	1.450844	18.063648	$0.16040150 \cdot 10^{-02}$	$0.62559307 \cdot 10^{-07}$

The values obtained in computer experiments are given in the tables. In the first column of the table, we indicate the number of interpolation nodes n. The second and third columns show, respectively, the values of b_{min} and b_{max}, and the fourth and fifth columns contain estimates E_1 and E_2.

Example 1. The function $y = 2^x$ is interpolated on $\mathcal{R} = [-0.45; 0.0]$. The derivative of the nth order of the function equals $y^{(n)} = 2^x(\ln 2)^n$. It is easy to see that, among derivatives of the order $k, k = \overline{n+1-r, n+1}$, the derivative of the $(n+1-r)$th order takes the highest value on the segment $\mathcal{R}$ on its right border, i.e., $f_{max} = (\ln 2)^{n+1-r}$. The results given by Table 9.1 show that estimate E_1 significantly concedes to estimate E_2. In addition, the values of E_2 decrease, as the number of interpolation nodes increases.

Example 2. Consider the problem of interpolation of the function $y = \sqrt{x}$ on $\mathcal{R} = [1{,}1; 1{,}7]$ T–ICF (2.4). Note that the nth derivative of the function is defined by the formula

$$\left(\sqrt{x}\right)^{(n)} = \frac{(-1)^{n+1}(2n-3)!!}{2^n \cdot \sqrt{x^{2n-1}}}.$$

Therefore, the derivative of the $(n+1)$th order takes the value highest in modulus among the derivatives of order k, with $k = \overline{n+1-r, n+1}$ on the left edge of the segment $\mathcal{R}$, i.e..,

$$f_{max} = \frac{(2n-1)!!}{2^{n+1}\left(\sqrt{1{,}1}\right)^{2n+1}},$$

The results presented in Table 9.2 show the advantage of estimate E_2 over estimate E_1. Like the previous example, the value of estimate E_2 decreases, as the number of interpolation nodes increases.

8. Conclusions

We have established new properties of a continuant which are used in studies of the problem of interpolation of functions of one real vatiable on a compact set by Thiele's interpolation continued fraction. A new form of the formula for the remainder term of T–ICF is obtained, and some estimates of the remainder term of T–ICF are made. The representation of the remainder term in terms of coninuants allows one to get other estimates of the remainder term of T–ICF.

The author sincerely thanks Academician of the NANU V. L. Makarov for valuable advices, useful remarks, and his interest in this work.

Table 9.2: Function $\sqrt{x}$, segment $\mathcal{R} = [1.1, 1.7]$

n	b_{min}	b_{max}	E_1	E_2
7	2.124195	2.585934	$0.66350373 \cdot 10^{-04}$	$0.21285556 \cdot 10^{-04}$
8	2.118777	2.590427	$0.48854910 \cdot 10^{-04}$	$0.11004596 \cdot 10^{-04}$
9	2.114851	2.593664	$0.38284594 \cdot 10^{-04}$	$0.58537385 \cdot 10^{-05}$
10	2.111918	2.596071	$0.31796009 \cdot 10^{-04}$	$0.31841731 \cdot 10^{-05}$
11	2.109671	2.597909	$0.27728209 \cdot 10^{-04}$	$0.17634902 \cdot 10^{-05}$
12	2.107913	2.599344	$0.25302938 \cdot 10^{-04}$	$0.99125665 \cdot 10^{-06}$
13	2.106512	2.600485	$0.24036410 \cdot 10^{-04}$	$0.56415945 \cdot 10^{-06}$
14	2.105378	2.601408	$0.23704991 \cdot 10^{-04}$	$0.32450711 \cdot 10^{-06}$
15	2.104447	2.602164	$0.24191062 \cdot 10^{-04}$	$0.18837630 \cdot 10^{-06}$
16	2.103673	2.602791	$0.25491251 \cdot 10^{-04}$	$0.11023125 \cdot 10^{-06}$
17	2.103024	2.603317	$0.27672961 \cdot 10^{-04}$	$0.64960423 \cdot 10^{-07}$
18	2.102474	2.603763	$0.30896745 \cdot 10^{-04}$	$0.38522821 \cdot 10^{-07}$
19	2.102011	2.604144	$0.35416462 \cdot 10^{-04}$	$0.22973370 \cdot 10^{-07}$
20	2.101615	2.604472	$0.41625056 \cdot 10^{-04}$	$0.13769827 \cdot 10^{-07}$
21	2.066126	2.604756	$0.65042318 \cdot 10^{-04}$	$0.85612464 \cdot 10^{-08}$

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Translated from Ukrainian by V.V. Kukhtin

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