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O. Pytovka

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A MODIFIED TWO-SIDED APPROXIMATION METHOD FOR A FOUR-POINT VALLÉE–POUSSIN TYPE PROBLEM

O. PYTOVKA

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Abstract. We develop a modified two-sided approximation method for a four-point boundary value problem of the Vallée–Poussin type for a system of non-linear differential equations of fourth order with argument deviations.

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Keywords: two-sided approximation method, Vallée–Poussin problem, differential inequality, comparison function

1. INTRODUCTION

There are many works dealing with constructive methods for approximate integration of boundary value problems for ordinary differential equations, which allow one to obtain a direct algorithm to error estimation (see, e. g., [4, 10, 11] and references therein). These methods include the two-sided methods, which give provide a possibility to construct approximate solutions and, on every step of iteration, obtain *a posteriori* error estimates of the successive approximations. Numerous research papers are devoted to the construction of new modifications of two-sided methods aimed at the study of various boundary value problems for ordinary differential equations (see, e. g., [1–3, 9].

This paper is devoted to the investigation of a four-point boundary-value problem of the Vallée–Poussin type for a system of non-linear differential equations with argument deviation by using a suitable version of the two-sided method generalising the works [5, 6].

2. PROBLEM SETTINGS, DEFINITIONS AND NOTATIONS

Let us consider the following problem of Vallée-Poussin's type: to find a solution $Y = (y_i)_{i=1}^n$ of the system of differential equations

$$Y^{(4)}(x) = F\left(x, Y(x), (\mathcal{J}_\Lambda Y)(x), (\mathcal{J}_\Theta Y)(x)\right), \quad x \in [0, \ell], \tag{2.1}$$

which satisfies the conditions

$$Y(0) = A_1, \quad Y(\ell/3) = A_2, \quad Y(2\ell/3) = A_3, \quad Y(\ell) = A_4, \tag{2.2}$$

and

$$Y(x) = \begin{cases} \Phi(x) & \text{if } x \in [\lambda_0, 0], \\ \Psi(x) & \text{if } x \in [\ell, \theta_0], \end{cases} \tag{2.3}$$

where $F\colon [0,\ell] \times \mathbb{R}^{3n} \to \mathbb{R}^n$, the vector-functions $\Lambda = (\lambda_i)_{i=1}^n$ and $\Theta = (\theta_i)_{i=1}^n$ from $C([0,\ell],\mathbb{R}^n)$* are such that $\lambda_i(x) \le x$, $\theta_i(x) \ge x$ for all $x \in [0,\ell]$, $i = \overline{1,n}$,

$$\lambda_0 := \min\{\lambda_i(x) \mid x \in [0,\ell],\ i = \overline{1,n}\}, \qquad \theta_0 := \max\{\theta_i(x) \mid x \in [0,\ell],\ i = \overline{1,n}\},$$

and $A_s = (a_{is})_{i=1}^n \in \mathbb{R}^n$ for $s = \overline{1,4}$, and $\Phi \in C([\lambda_0, 0],\mathbb{R}^n)$, $\Psi \in C([\ell,\theta_0],\mathbb{R}^n)$ are given initial vector-functions satisfying the conditions

$$\Phi(0) = A_1, \quad \Psi(\ell) = A_4. \tag{2.4}$$

The operator $\mathcal{J}_\Gamma\colon C([\lambda_0,\theta_0],\mathbb{R}^n) \to C([0,\ell],\mathbb{R}^n)$ appearing in (2.1) is defined by the formula

$$(\mathcal{J}_\Gamma Y)(x) := \big(y_i(\gamma_i(x))\big)_{i=1}^n, \quad x \in [0,\ell],$$

for any $\Gamma = (\gamma_i)_{i=1}^n \in C([0,\ell],\mathbb{R}^n)$ and $Y = (y_i)_{i=1}^n \in C([\lambda_0,\theta_0],\mathbb{R}^n)$.

3. Assumptions

In the sequel, let us suppose that the right-hand side $F\colon [0,\ell] \times \mathcal{D}^3 \to \mathbb{R}^n$, $\mathcal{D} \subseteq \mathbb{R}^n$, of the equation (2.1) belongs to the class $\mathcal{M}_{\mathcal{D}}([0,\ell])$, where $\mathcal{M}_{\mathcal{D}}([0,\ell])$ denotes the set of the vector-functions F satisfying the following conditions:

(1) $F \in C([0,\ell] \times \mathcal{D}^3, \mathbb{R}^n)$;

(2) there exists a vector-function $H \in C([0,\ell] \times \mathcal{D}^6, \mathbb{R}^n)$ such that:

  (a) the equality

$$H(x,U,U) = F(x,U)$$

  holds for all $x \in [0,\ell]$ and $U \in \mathcal{D}^3$;

  (b) the inequality

$$\begin{aligned} &H\big(x, P_1(x), (\mathcal{J}_\Lambda P_1)(x), (\mathcal{J}_\Theta P_1)(x), Q_2(x), (\mathcal{J}_\Lambda Q_2)(x), (\mathcal{J}_\Theta Q_2)(x)\big) \\ &\ge H\big(x, Q_1(x), (\mathcal{J}_\Lambda Q_1)(x), (\mathcal{J}_\Theta Q_1)(x), P_2(x), (\mathcal{J}_\Lambda P_2)(x), (\mathcal{J}_\Theta P_2)(x)\big) \end{aligned} \tag{3.1}$$

  is satisfied for all $x \in [0,\ell]$ and every vector-functions $P_k, Q_k\colon [\tau_0,\theta_0] \to \mathbb{R}^n$, $k = 1,2$, whose restrictions on $[0,\ell]$ belong to $C^4([0,\ell],\mathbb{R}^n)$, such that $P_k(x), Q_k(x) \in \mathcal{D}$ for all $x \in [\lambda_0,\theta_0]$, $k = 1,2$, and

$$\begin{aligned} P_k(x) &\le Q_k(x) && \text{for } x \in [0,\ell/3] \cup [2\ell/3,\ell],\ k = 1,2, \\ P_k(x) &\ge Q_k(x) && \text{for } x \in [\ell/3, 2\ell/3],\ k = 1,2, \\ P_k^{(4)}(x) &\ge Q_k^{(4)}(x) && \text{for } x \in [0,\ell],\ k = 1,2. \end{aligned}$$

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*$C([0,\ell],\mathbb{R}^n)$ is the usual Banach space of continuous vector-functions from $[0,\ell]$ to $\mathbb{R}^n$.

(c) the vector-function H satisfies the Lipschitz condition with a non-negative matrix $K=(k_{ij})_{i,j=1}^{n}$, i. e.,

$$|H(x,P_{10},P_{11},P_{12},Q_{10},Q_{11},Q_{12})-H(x,P_{00},P_{01},P_{02},Q_{00},Q_{01},Q_{02})| \le K\left(\sum_{s=0}^{2}(|P_{1s}-P_{0s}|+|Q_{1s}-Q_{0s}|)\right), \quad (3.2)$$

for all $P_{s0},P_{s1},P_{s2},Q_{s0},Q_{s1},Q_{s2}$ from $\mathcal{D}$, $s=0,1$, and all $x\in[0,\ell]$.

In (3.1), (3.2), and all similar relations below, the inequalities between vectors and the absolute value sign are understood component-wise.

4. Preliminary considerations

Due to the fact that the corresponding linearised homogeneous boundary value problem has only the trivial solution on $[0,\ell]$, the solution Y of problem (2.1)–(2.3) can be represented in the form

$$Y(x)=\begin{cases}\Phi(x) & \text{for } x\in[\lambda_0,0],\\ \Omega(x)-(\mathcal{T}F(\cdot,Y(\cdot),(\mathcal{J}_{\Lambda}Y)(\cdot),(\mathcal{J}_{\Theta}Y)(\cdot)))(x) & \text{for } x\in[0,\ell],\\ \Psi(x) & \text{for } x\in[\ell,\theta_0],\end{cases} \quad (4.1)$$

where the vector-function $\Omega(x)=(\omega_i(x))_{i=1}^{n}$ has the components

$$\omega_i(x)=a_{i1}+\frac{243}{4\ell^6}\begin{vmatrix} x & 0 & x^2 & x^3\\ \frac{\ell}{3} & a_{i2}-a_{i1} & \frac{\ell^2}{9} & \frac{\ell^3}{27}\\ \frac{2\ell}{3} & a_{i3}-a_{i1} & \frac{4\ell^2}{9} & \frac{8\ell^3}{27}\\ \ell & a_{i4}-a_{i1} & \ell^2 & \ell^3\end{vmatrix}, \quad x\in[0,\ell],$$

the operator $\mathcal{T}\colon C([0,\ell],\mathbb{R}^n)\to C([0,\ell],\mathbb{R}^n)$ for any $Z\in C([0,\ell],\mathbb{R}^n)$ is defined by the formula

$$(\mathcal{T}Z)(x):=\frac{81}{8\ell^6}\int_0^{\ell}\mathcal{G}(x,\xi)Z(\xi)d\xi, \quad x\in[0,\ell],$$

and $\mathcal{G}$ is the Green function [7,8] of the problem given by the relations

$$\mathcal{G}(x,\xi)=\begin{cases}\mathcal{G}_1(x,\xi), & 0\le x\le\frac{\ell}{3},\\ \mathcal{G}_2(x,\xi), & \frac{\ell}{3}\le x\le\frac{2\ell}{3},\\ \mathcal{G}_3(x,\xi), & \frac{2\ell}{3}\le x\le\ell,\end{cases} \qquad \mathcal{G}_1(x,\xi)=\begin{cases}R_{11}(x,\xi), & 0\le\xi\le x,\\ R_{12}(x,\xi), & x\le\xi\le\frac{\ell}{3},\\ R_{13}(x,\xi), & \frac{\ell}{3}\le\xi\le\frac{2\ell}{3},\\ R_{14}(x,\xi), & \frac{2\ell}{3}\le\xi\le\ell,\end{cases}$$

$$\mathcal{G}_2(x,\xi)=\begin{cases}R_{21}(x,\xi), & 0\le\xi\le\frac{\ell}{3},\\ R_{22}(x,\xi), & \frac{\ell}{3}\le\xi\le x,\\ R_{23}(x,\xi), & x\le\xi\le\frac{2\ell}{3},\\ R_{24}(x,\xi), & \frac{2\ell}{3}\le\xi\le\ell,\end{cases} \qquad \mathcal{G}_3(x,\xi)=\begin{cases}R_{31}(x,\xi), & 0\le\xi\le\frac{\ell}{3},\\ R_{32}(x,\xi), & \frac{\ell}{3}\le\xi\le\frac{2\ell}{3},\\ R_{33}(x,\xi), & \frac{2\ell}{3}\le\xi\le x,\\ R_{34}(x,\xi), & x\le\xi\le\ell,\end{cases}$$

$$R_{k1}(x,\xi)=\begin{vmatrix} x & (x-\xi)^3 & x^2 & x^3 \\ \frac{\ell}{3} & (\frac{\ell}{3}-\xi)^3 & \frac{\ell^2}{9} & \frac{\ell^3}{27} \\ \frac{2\ell}{3} & (\frac{2\ell}{3}-\xi)^3 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27} \\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix}, \; R_{k4}(x,\xi)=\begin{vmatrix} x & 0 & x^2 & x^3 \\ \frac{\ell}{3} & 0 & \frac{\ell^2}{9} & \frac{\ell^3}{27} \\ \frac{2\ell}{3} & 0 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27} \\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix}$$

for $k=\overline{1,3}$,

$$R_{12}(x,\xi)=\begin{vmatrix} x & 0 & x^2 & x^3 \\ \frac{\ell}{3} & (\frac{\ell}{3}-\xi)^3 & \frac{\ell^2}{9} & \frac{\ell^3}{27} \\ \frac{2\ell}{3} & (\frac{2\ell}{3}-\xi)^3 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27} \\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix}, \; R_{33}(x,\xi)=\begin{vmatrix} x & (x-\xi)^3 & x^2 & x^3 \\ \frac{\ell}{3} & 0 & \frac{\ell^2}{9} & \frac{\ell^3}{27} \\ \frac{2\ell}{3} & 0 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27} \\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix},$$

and

$$R_{22}(x,\xi)=R_{32}(x,\xi)=\begin{vmatrix} x & (x-\xi)^3 & x^2 & x^3 \\ \frac{\ell}{3} & 0 & \frac{\ell^2}{9} & \frac{\ell^3}{27} \\ \frac{2\ell}{3} & (\frac{2\ell}{3}-\xi)^3 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27} \\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix},$$

$$R_{13}(x,\xi)=R_{23}(x,\xi)=\begin{vmatrix} x & 0 & x^2 & x^3 \\ \frac{\ell}{3} & 0 & \frac{\ell^2}{9} & \frac{\ell^3}{27} \\ \frac{2\ell}{3} & (\frac{2\ell}{3}-\xi)^3 & \frac{4\ell^2}{9} & \frac{8\ell^3}{27} \\ \ell & (\ell-\xi)^3 & \ell^2 & \ell^3 \end{vmatrix}.$$

It is easy to see that

$$\mathcal{G}_1(x,\xi)\geq 0, \quad \mathcal{G}_2(x,\xi)\leq 0, \quad \mathcal{G}_3(x,\xi)\geq 0 \quad \text{for } (x,\xi)\in[0,\ell]\times[0,\ell]. \tag{4.2}$$

Definition. Vector-functions $Z_0, V_0\colon[\lambda_0,\theta_0]\to\mathcal{D}$ whose restrictions on $[0,\ell]$ belong to the space $C^4([0,\ell],\mathbb{R}^n)$ are called *comparison functions of problem* (2.1)–(2.3) if they satisfy the boundary conditions (2.2), the initial condition (2.3), and the inequalities

$$\begin{aligned} Z_0(x)&\leq V_0(x) \quad \text{for } x\in[0,\ell/3]\cup[2\ell/3,\ell], \\ Z_0(x)&\geq V_0(x) \quad \text{for } x\in[\ell/3,2\ell/3]. \end{aligned} \tag{4.3}$$

Notation. For any vector-functions $P,Q\colon[\lambda_0,\theta_0]\to\mathbb{R}^n$ we set

$$\langle P,Q\rangle=\big\{u\in\mathbb{R}^n \mid \min\{P(x),Q(x)\}\leq u\leq\max\{P(x),Q(x)\} \text{ for some } x\in[\lambda_0,\theta_0]\big\},$$

where the operations "min" and "max" for vectors are understood component-wise.

5. Construction of the alternative two-sided method for problem (2.1)–(2.3)

Let us construct the successive approximations $\{Z_p\}_{p=1}^{\infty}$ and $\{V_p\}_{p=1}^{\infty}$ of a solution of problem (2.1)–(2.3) according to the formulae

$$Z_{p+1}(x)=\begin{cases}\Phi(x) & \text{for } x\in[\lambda_0,0],\\ \Omega(x)-(\mathcal{T}F_p)(x) & \text{for } x\in[0,\ell],\\ \Psi(x) & \text{for } x\in[\ell,\theta_0],\end{cases}$$
$$V_{p+1}(x)=\begin{cases}\Phi(x) & \text{for } x\in[\lambda_0,0],\\ \Omega(x)-(\mathcal{T}F^p)(x) & \text{for } x\in[0,\ell],\\ \Psi(x) & \text{for } x\in[\ell,\theta_0],\end{cases} \tag{5.1}$$

where

$$F^p(x)=H\big(x,Z_p(x),(\mathcal{J}_\Lambda Z_p)(x),(\mathcal{J}_\Theta Z_p)(x),V_p(x),(\mathcal{J}_\Lambda V_p)(x),(\mathcal{J}_\Theta V_p)(x)\big),$$
$$F_p(x)=H\big(x,V_p(x),(\mathcal{J}_\Lambda V_p)(x),(\mathcal{J}_\Theta V_p)(x),Z_p(x),(\mathcal{J}_\Lambda Z_p)(x),(\mathcal{J}_\Theta Z_p)(x)\big)$$

for all $x\in[0,\ell]$, and the zero approximations Z_0 and V_0 are comparison functions of problem (2.1)–(2.3) satisfying the conditions

$$\alpha_0(x):=Z_0^{(4)}(x)-F_0(x)\geq 0,$$
$$\beta_0(x):=V_0^{(4)}(x)-F^0(x)\leq 0 \tag{5.2}$$

for all $x\in[0,\ell]$.

The iteration process (5.1) can be represented in the form

$$Z_{p+1}(x)-Z_p(x)=(\mathcal{T}\alpha_p)(x),\quad V_{p+1}(x)-V_p(x)=(\mathcal{T}\beta_p)(x),\quad x\in[0,\ell], \tag{5.3}$$

where

$$\alpha_p(x):=Z_p^{(4)}(x)-F_p(x),\quad \beta_p(x):=V_p^{(4)}(x)-F^p(x),\quad x\in[0,\ell],\ p\in\mathbb{N}. \tag{5.4}$$

Hence, from (5.3) and (5.4), for any $p\in\mathbb{N}\cup\{0\}$, we obtain

$$\alpha_{p+1}(x)=F_p(x)-F_{p+1}(x),\quad \beta_{p+1}(x)=F^p(x)-F^{p+1}(x),\quad x\in[0,\ell], \tag{5.5}$$

$$Z_p(x)-Z_{p+2}(x)=-\mathcal{T}(\alpha_p+\alpha_{p+1})(x),\quad x\in[0,\ell],$$
$$V_p(x)-V_{p+2}(x)=-\mathcal{T}(\beta_p+\beta_{p+1})(x),\quad x\in[0,\ell], \tag{5.6}$$

and

$$\alpha_{p+1}(x)+\alpha_{p+2}(x)=F_p(x)-F_{p+2}(x),\quad x\in[0,\ell],$$
$$\beta_{p+1}(x)+\beta_{p+2}(x)=F^p(x)-F^{p+2}(x),\quad x\in[0,\ell]. \tag{5.7}$$

Taking into account conditions (4.2), (5.2), and (5.3) with $p=0$, we can see that

$$Z_1(x)-Z_0(x)\geq 0,\quad V_1(x)-V_0(x)\leq 0,\quad x\in[0,\ell/3]\cup[2\ell/3,\ell],$$
$$Z_1(x)-Z_0(x)\leq 0,\quad V_1(x)-V_0(x)\geq 0,\quad x\in[\ell/3,2\ell/3]. \tag{5.8}$$

Thus, if $Z_1(x), V_1(x) \in \mathcal{D}$ for all $x \in [\lambda_0, \theta_0]$, then from (5.5) with $p = 0$, by virtue of (5.2), (5.8), and (3.1), we obtain $\alpha_1(x) \leq 0$, $\beta_1(x) \geq 0$ for all $x \in [0, \ell]$. Therefore, from (4.2) and (5.3) with $p = 1$ we get

$$\begin{aligned} &Z_2(x) - Z_1(x) \leq 0, \quad V_2(x) - V_1(x) \geq 0, \qquad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_2(x) - Z_1(x) \geq 0, \quad V_2(x) - V_1(x) \leq 0, \qquad x \in [\ell/3, 2\ell/3]. \end{aligned} \tag{5.9}$$

Assume, in addition, that

$$\alpha_0(x) + \alpha_1(x) \geq 0, \quad \beta_0(x) + \beta_1(x) \leq 0, \quad x \in [0, \ell]. \tag{5.10}$$

Then from (5.6) with $p = 0$ we obtain

$$\begin{aligned} &Z_0(x) - Z_2(x) \leq 0, \quad V_0(x) - V_2(x) \geq 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_0(x) - Z_2(x) \geq 0, \quad V_0(x) - V_2(x) \leq 0, \quad x \in [\ell/3, 2\ell/3], \end{aligned} \tag{5.11}$$

and thus (5.9) and (5.11) result in

$$Z_0(x) \leq Z_2(x) \leq Z_1(x), \quad V_1(x) \leq V_2(x) \leq V_0(x), \quad \text{for } x \in [0, \ell/3] \cup [2\ell/3, \ell], \tag{5.12}$$

and

$$Z_1(x) \leq Z_2(x) \leq Z_0(x), \quad V_0(x) \leq V_2(x) \leq V_1(x) \quad \text{for } x \in [\ell/3, 2\ell/3]. \tag{5.13}$$

Therefore, we have proved that if $\langle Z_0, Z_1 \rangle \subseteq \mathcal{D}$, $\langle V_1, V_0 \rangle \subseteq \mathcal{D}$, and conditions (5.10) hold, then the values $Z_2(x)$ and $V_2(x)$ of the next approximations which are obtained according to (5.1) also belong to the set $\mathcal{D}$.

From (3.1), (5.10), (5.12), (5.13), and (5.3), (5.5), (5.7) with $p = 2, 1, 0$, we get

$$\begin{aligned} &\alpha_2(x) \geq 0, \quad \beta_2(x) \leq 0, \quad x \in [0, \ell], \\ &Z_3(x) - Z_2(x) \geq 0, \quad V_3(x) - V_2(x) \leq 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_3(x) - Z_2(x) \leq 0, \quad V_3(x) - V_2(x) \geq 0, \quad x \in [\ell/3, 2\ell/3], \end{aligned}$$

and

$$\alpha_1(x) + \alpha_2(x) \leq 0, \quad \beta_1(x) + \beta_2(x) \geq 0, \quad x \in [0, \ell].$$

Hence, from (5.6) with $p = 1$ we obtain

$$\begin{aligned} &Z_1(x) - Z_3(x) \geq 0, \quad V_1(x) - V_3(x) \leq 0, \quad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_1(x) - Z_3(x) \leq 0, \quad V_1(x) - V_3(x) \geq 0, \quad x \in [\ell/3, 2\ell/3]. \end{aligned}$$

Consequently,

$$\begin{aligned} &Z_0(x) \leq Z_2(x) \leq Z_3(x) \leq Z_1(x), \quad V_1(x) \leq V_3(x) \leq V_2(x) \leq V_0(x), \\ &\qquad x \in [0, \ell/3] \cup [2\ell/3, \ell], \\ &Z_1(x) \leq Z_3(x) \leq Z_2(x) \leq Z_0(x), \quad V_0(x) \leq V_2(x) \leq V_3(x) \leq V_1(x), \\ &\qquad x \in [\ell/3, 2\ell/3], \end{aligned}$$

and thus $Z_3(x), V_3(x) \in \mathcal{D}$ for all $x \in [\lambda_0, \theta_0]$.

Using the method of the mathematical induction we can show that if $\langle Z_0, Z_1\rangle \subseteq \mathcal{D}$, $\langle V_1, V_0\rangle \subseteq \mathcal{D}$, and conditions (5.10) hold, then the sequences $\{Z_p\}_{p=1}^{\infty}$ and $\{V_p\}_{p=1}^{\infty}$, which are constructed according to (5.1), satisfy the inequalities

$$Z_{2p}(x) \leq Z_{2p+2}(x) \leq Z_{2p+3}(x) \leq Z_{2p+1}(x),$$
$$V_{2p+1}(x) \leq V_{2p+3}(x) \leq V_{2p+2}(x) \leq V_{2p}(x)$$

for $x \in [0, \ell/3] \cup [2\ell/3, \ell]$, $p = 0, 1, 2, \ldots$, and

$$Z_{2p+1}(x) \leq Z_{2p+3}(x) \leq Z_{2p+2}(x) \leq Z_{2p}(x),$$
$$V_{2p}(x) \leq V_{2p+2}(x) \leq V_{2p+3}(x) \leq V_{2p+1}(x)$$

for $x \in [\ell/3, 2\ell/3]$, $p = 0, 1, 2, \ldots$.

Let us now find a sufficient condition for the uniform, on $[\lambda_0, \theta_0]$, convergence of the sequences $\{Z_p\}_{p=1}^{\infty}$ and $\{V_p\}_{p=1}^{\infty}$ to the unique solution of the boundary value problem (2.1)–(2.3).

For any vector $P = (p_i)_{i=1}^{n} \in \mathbb{R}^n$, we set

$$\|P\| := \max_{i=\overline{1,n}} |p_i|.$$

Let us also put

$$W_p(x) := Z_p(x) - V_p(x), \quad x \in [\lambda_0, \theta_0],\ p = 0, 1, 2 \ldots,$$

$$\epsilon := \max_{x\in[0,\ell]} \Big\{ \|Z_0(x) - Z_1(x)\|, \|V_0(x) - V_1(x)\|, \|W_0(x)\| \Big\},$$

and

$$d := \max_{x\in[0,\ell]} \int_0^{\ell} |\mathcal{G}(x,\xi)| d\xi = \frac{4\ell^{10}}{3^7}.$$

Then using (5.3), (5.5), we can prove by induction the error estimate

$$\max_{x\in[0,\ell]} \Big\{ \|Z_{p+1}(x) - Z_p(x)\|, \|V_{p+1}(x) - V_p(x)\| \Big\} \leq \epsilon \left(\frac{81}{8\ell^6} d6\|K\| \right)^p = \epsilon \left(\frac{\ell^4}{9} \|K\| \right)^p \tag{5.14}$$

valid for all $p \in \mathbb{N}$, where K is the matrix appearing in the Lipschitz condition (3.2) and $\|K\| = \max_{i=\overline{1,n}} \{\sum_{j=1}^{n} k_{ij}\}$.

If $\|K\|$ satisfies the inequality

$$\|K\| < \frac{9}{\ell^4}, \tag{5.15}$$

then it follows from estimate (5.14) that the approximations $\{Z_p\}_{p=1}^{\infty}$ and $\{V_p\}_{p=1}^{\infty}$ converge, respectively, to certain limits Y_* and Y^* uniformly on $[\lambda_0, \theta_0]$.

Let us show that $Y_*(x) \equiv Y^*(x)$. From (5.1) we have

$$W_{p+1}(x) = \begin{cases} 0 & \text{for } x \in [\lambda_0, 0], \\ \big(\mathcal{T}(F^p - F_p)\big)(x) & \text{for } x \in [0, \ell], \\ 0 & \text{for } x \in [\ell, \theta_0]. \end{cases}$$

It is easy to show that the estimate

$$\max_{x\in[0,\ell]} \|W_p(x)\| \le \epsilon \left(\frac{81}{8\ell^6} d6\|K\|\right)^p = \epsilon \left(\frac{\ell^4}{9}\|K\|\right)^p \tag{5.16}$$

is true for $p \in \mathbb{N}$. If condition (5.15) holds, then $\lim_{p\to\infty} W_p(x) = 0$ uniformly on $[0, \ell]$, and thus

$$Y_*(x) = Y^*(x) =: Y(x), \quad x \in [\lambda_0, \theta_0].$$

Passing in equalities (5.1) to the limit as $p \to \infty$, we obtain the equality

$$Y(x) = \begin{cases} \Phi(x) & \text{for } x \in [\lambda_0, 0], \\ \Omega(x) - \big(\mathcal{T}\widetilde{H}\big)(x) & \text{for } x \in [0, \ell], \\ \Psi(x) & \text{for } x \in [\ell, \theta_0], \end{cases}$$

where

$$\widetilde{H}(x) := H\big(x, Y(x), (\mathcal{J}_\Lambda Y)(x), (\mathcal{J}_\Theta Y)(x), Y(x), (\mathcal{J}_\Lambda Y)(x), (\mathcal{J}_\Theta Y)(x)\big)$$
$$= F\big(x, Y(x), (\mathcal{J}_\Lambda Y)(x), (\mathcal{J}_\Theta Y)(x)\big), \quad x \in [0, \ell],$$

i. e., Y is a solution of problem (2.1)–(2.3).

The uniqueness of the solution Y under the condition (5.15) can be easily proved by using the Lipschitz condition (3.2).

Consequently, we have proved the following

Theorem. *Let $F \in \mathcal{M}_{\mathfrak{D}}([0,\ell])$ and Z_0, V_0 be comparison functions of problem (2.1)–(2.3) satisfying conditions (5.2). In addition, let the first approximations Z_1 and V_1 constructed according to formulae (5.1) be such that $\langle Z_0, Z_1\rangle \subseteq \mathfrak{D}$, $\langle V_1, V_0\rangle \subseteq \mathfrak{D}$, and conditions (5.10) hold. Assume also that condition (5.15) is satisfied.*

Then the sequences of approximations $\{Z_p\}_{p=1}^\infty$ and $\{V_p\}_{p=1}^\infty$ constructed according to (5.1) converge uniformly on $[\lambda_0, \theta_0]$ to the unique solution Y of problem (2.1)–(2.3) and, moreover,

$$Z_{2p}(x) \le Z_{2p+2}(x) \le Y(x) \le Z_{2p+3}(x) \le Z_{2p+1}(x),$$
$$V_{2p+1}(x) \le V_{2p+3}(x) \le Y(x) \le V_{2p+2}(x) \le V_{2p}(x)$$

for $x \in [0, \ell/3] \cup [2\ell/3, \ell]$, $p = 0, 1, 2, \ldots$, and

$$Z_{2p+1}(x) \le Z_{2p+3}(x) \le Y(x) \le Z_{2p+2}(x) \le Z_{2p}(x),$$

$$V_{2p}(x) \le V_{2p+2}(x) \le Y(x) \le V_{2p+3}(x) \le V_{2p+1}(x)$$

for $x \in [\ell/3, 2\ell/3]$, $p = 0, 1, 2, \ldots$.

Remark. If the domain $\mathcal{D}$ is "large" enough, then there exist comparison functions Z_0, V_0 of problem (2.1)–(2.3) satisfying conditions (5.2).

Indeed, let $U\colon [\lambda_0, \theta_0] \to \mathbb{R}^n$ be an arbitrary vector-function which satisfies the boundary conditions (2.2) and the initial condition (2.3) and is such that $U|_{[0,\ell]} \in C^4([0,\ell], \mathbb{R}^n)$ and $U(x) \in \mathcal{D}$ for all $x \in [\lambda_0, \theta_0]$. Then we set

$$\alpha(x) := U^{(4)}(x) - F\left(x, U(x), (\mathcal{J}_\Lambda U)(x), (\mathcal{J}_\Theta U)(x)\right), \quad x \in [0,\ell]. \tag{5.17}$$

It is clear that the problems

$$\eta^{(4)} = |\alpha(x)|,$$
$$\eta(0) = 0, \quad \eta(\ell/3) = 0, \quad \eta(2\ell/3) = 0, \quad \eta(\ell) = 0$$

and

$$q^{(4)} = -|\alpha(x)|,$$
$$q(0) = 0, \quad q(\ell/3) = 0, \quad q(2\ell/3) = 0, \quad q(\ell) = 0$$

have unique solutions η and q, respectively. Relations (4.1) and (4.2) yield

$$\begin{aligned} &\eta(x) \le 0, \quad q(x) \ge 0, \quad x \in [0,\ell/3] \cup [2\ell/3, \ell], \\ &\eta(x) \ge 0, \quad q(x) \le 0, \quad x \in [\ell/3, 2\ell/3]. \end{aligned} \tag{5.18}$$

Now we put

$$\begin{aligned} &Z_0(x) = U(x) + \eta(x), \quad V_0(x) = U(x) + q(x), \quad x \in [0,\ell], \\ &Z_0(x) = U(x), \qquad\qquad V_0(x) = U(x), \qquad\qquad x \in [\lambda_0, 0] \cup [\ell, \theta_0]. \end{aligned}$$

It is easy to see that Z_0 and V_0 satisfy the boundary conditions (2.2), the initial condition (2.3), and inequalities (4.3). If $Z_0(x), V_0(x) \in \mathcal{D}$ for all $x \in [\lambda_0, \theta_0]$, then Z_0, V_0 are comparison functions of problem (2.1)–(2.3) and, using (5.17), (5.18) and assumptions (2a) and (2b) of Section 3, we get

$$\begin{aligned} Z_0^{(4)}(x) - F_0(x) &= U^{(4)}(x) + |\alpha(x)| - F_0(x) = \\ &= \alpha(x) + |\alpha(x)| + F\left(x, U(x), (\mathcal{J}_\Lambda U)(x), (\mathcal{J}_\Theta U)(x)\right) - F_0(x) \ge 0 \end{aligned}$$

and

$$\begin{aligned} V_0^{(4)}(x) - F^0(x) &= U^{(4)}(x) - |\alpha(x)| - F^0(x) = \\ &= \alpha(x) - |\alpha(x)| + F\left(x, U(x), (\mathcal{J}_\Lambda U)(x), (\mathcal{J}_\Theta U)(x)\right) - F^0(x) \le 0 \end{aligned}$$

for all $x \in [0,\ell]$. Consequently, Z_0 and V_0 also satisfy conditions (5.2).

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Author's address

O. Pytovka
State University of Mukachevo, 26 Uzhgorodska St., Mukachevo, Ukraine
E-mail address: oxana_pityovka@bigmir.net