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A CONTINUANT AND AN ESTIMATE OF THE REMAINDER OF THE INTERPOLATING CONTINUED C-FRACTION

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The problem of the interpolation of functions of a real variable by interpolating continued C-fraction is investigated. The relationship between the continued fraction and the continuant was used. The properties of the continuant are established. The formula for the remainder of the interpolating continued C-fraction proved. The remainder expressed in terms of derivatives of the functional continent. An estimate of the remainder was obtained. The main result of this paper is contained in the following Theorem 5:

Let $\mathcal{R} \subset \mathbb{R}$ be a compact, $f \in \mathbf{C}^{(n+1)}(\mathcal{R})$ and the interpolating continued C-fraction (C-ICF) of the form

$$D_n(x) = \frac{P_n(x)}{Q_n(x)} = a_0 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k(x - x_{k-1})}{1}, \ a_k \in \mathbb{R}, \ k = \overline{0,n},$$

be constructed by the values the function f at nodes $X = \{x_i : x_i \in \mathcal{R}, x_i \neq x_j, i \neq j, i, j = \overline{0,n}\}$. If the partial numerators of C-ICF satisfy the condition of the Paydon–Wall type, that is $0 < a^* \operatorname{diam} \mathcal{R} \leq p$, then

$$|f(x) - D_n(x)| \leq \frac{f^* \prod\limits_{k=0}^{n} |x - x_k|}{(n+1)!\,\Omega_n(t)} \Big(\kappa_{n+1}(p) + \sum_{k=1}^{r} \tbinom{n+1}{k} (a^*)^k \sum_{i_1=1}^{n+1-2k} \kappa_{i_1}(p) \times$$
$$\times \sum_{i_2=i_1+2}^{n+3-3k} \kappa_{i_2-i_1-1}(p) \cdots \sum_{i_{k-1}=i_{k-2}+2}^{n-3} \kappa_{i_{k-1}-i_{k-2}-1}(p) \sum_{i_k=i_{k-1}+2}^{n-1} \kappa_{i_k-i_{k-1}-1}(p)\, \kappa_{n-i_k}(p) \Big),$$

where $f^* = \max\limits_{0 \leq m \leq r} \max\limits_{x \in \mathcal{R}} |f^{(n+1-m)}(x)|$, $\kappa_n(p) = \dfrac{(1+\sqrt{1+4p})^n - (1-\sqrt{1+4p})^n}{2^n \sqrt{1+4p}}$, $a^* = \max\limits_{2 \leqslant i \leqslant n} |a_i|$, $p = t(1-t)$, $t \in (0; \frac{1}{2}]$, $r = \left[\frac{n}{2}\right]$.

1. Introduction. The need to interpolate a function of the one real variable arises, as an auxiliary task, in solving many problems of mathematics, applied mathematics, physics, mechanics, engineering, economics, etc. The problem of interpolation has independent importance too. The functions can be interpolated by the polynomials ([1, 2, 3]), splines ([4]), rational functions ([2]), Padé approximants ([5]), etc.

In addition to these methods, the function f defined on the compact $\mathcal{R} \subset \mathbb{R}$ can be interpolated by different types of continued fractions ([6]). An estimate of the remainder of

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the interpolating continued C-fraction was obtained in [7]. The current article is devoted to obtaining new properties of the continuant and to prove on their basis an estimate of the remainder of the interpolating continued C-fraction.

2. interpolating continued C-fraction. Here are necessary definitions, formulas, statements in the theory of fractions ([8]). Let $b_0, a_k \neq 0, b_k, k \in \mathbb{N}$, be numbers, functions, functionals, matrices, operators, and so on. Infinite continued fraction of the form

$$D = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}} \tag{1}$$

will be briefly written as follows

$$D = b_0 + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots + \frac{a_k}{b_k +} \cdots = b_0 + \mathbf{K}(a_k/b_k).$$

Similarly, nth approximation of the infinite continued fraction (1) is briefly written as

$$D_n = \frac{P_n}{Q_n} = b_0 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots + \frac{a_n}{b_n} = b_0 + \mathbf{K}_{k=1}^{n}(a_k/b_k). \tag{2}$$

The quantities $P_n, Q_n, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, are called the canonical numerator and denominator, and $b_0, a_k, b_k, k \in \mathbb{N}$, are called elements of the continued fraction. The values of P_n and Q_n can be determined by the elements of the continued fraction using Wallis formulas

$$\begin{aligned} P_n &= b_n P_{n-1} + a_n P_{n-2}, \qquad P_{-1} = 1,\ P_0 = b_0. \\ Q_n &= b_n Q_{n-1} + a_n Q_{n-2}, \qquad Q_{-1} = 0,\ Q_0 = 1, \quad n \in \mathbb{N}. \end{aligned} \tag{3}$$

Definition 1. Two continued fractions $b_0 + \mathbf{K}(a_k/b_k)$ and $d_0 + \mathbf{K}(c_k/d_k)$ are said to be *equivalent* if and only if they have the same sequence of approximants, $b_0 + \mathbf{K}_{k=1}^{n}(a_k/b_k) \equiv d_0 + \mathbf{K}_{k=1}^{n}(c_k/d_k)$.

Theorem 1 ([8])**.** *Continued fractions $b_0 + \mathbf{K}(a_k/b_k)$ and $d_0 + \mathbf{K}(c_k/d_k)$ are equivalent if and only if there exists a sequence of non-zero constants $\{r_k : r_0 = 1, r_k \neq 0, k \in \mathbb{N}\}$ such as*

$$d_0 = b_0,\ c_k = r_{k-1} r_k a_k,\ d_k = r_k b_k,\ k \in \mathbb{N}. \tag{4}$$

Consider the problem of interpolation of the functions by a continued fraction. Let the set of interpolation nodes be selected on the compact $\mathcal{R} \subset \mathbb{R}$

$$X = \{x_i : x_i \in \mathcal{R}, x_i \neq x_j, i \neq j, i, j = \overline{0, n}\}. \tag{5}$$

The function f will be interpolated by the continued fraction of the form

$$D_n(x) = \frac{P_n(x)}{Q_n(x)} = a_0 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k(x - x_{k-1})}{1}, \qquad a_k \in \mathbb{R},\ k = \overline{0, n}. \tag{6}$$

A continued fraction (6) is called an *interpolating continued C-fraction* (C-ICF) ([6]).

The C-ICF satisfies the interpolation conditions $D_n(x_k) = y_k,\ x_k \in X,\ y_k = f(x_k)$, $k = \overline{0,n}$. Its coefficients $a_k, k \in \mathbb{N}_0$, are determined by the following recurrence relation in the form of a continued fraction

$$a_0 = y_0,\ a_1 = \frac{y_1 - y_0}{x_1 - x_0},\ a_k = \frac{1}{x_k - x_{k-1}}\Bigg(-1 + \frac{a_{k-1}(x_k - x_{k-2})}{-1} + \frac{a_{k-2}(x_k - x_{k-3})}{-1} + \\ + \frac{a_{k-3}(x_k - x_{k-4})}{-1} + \cdots + \frac{a_2(x_k - x_1)}{-1} + \frac{a_1(x_k - x_0)}{y_k - y_0}\Bigg), \qquad k = \overline{2,n}. \tag{7}$$

It is easy to see, that C-ICF (6) is a rational function. The degree of the polynomials of numerator and denominator satisfy the inequalities $\deg P_n(x) \le \left[\frac{n+1}{2}\right], \deg Q_n(x) \le \left[\frac{n}{2}\right]$. It is well known that C-ICF is equivalent to the Thiele interpolating continued fraction ([9]).

Theorem 2 ([7])**.** *Suppose the function $f \in \mathbf{C}^{(n+1)}(\mathcal{R})$. Let C-ICF (6) be constructed by the values of the function f at nodes (5). Then the remainder of the C-ICF satisfies the inequality*

$$\left| f(x) - \frac{P_n(x)}{Q_n(x)} \right| \le \frac{f^* \prod_{k=0}^n |x - x_k|}{(n+1)!\,|Q_n(x)|} \left(\kappa_{n+1}(\rho) + \sum_{m=1}^{r} \binom{n+1}{m} (a^*)^m \sum_{i=0}^{r-m} \frac{\rho^i}{i!} \prod_{j=1}^{m+i} (n - 2(m+i) + j) \right),$$

where

$$r = [n/2],\ f^* = \max_{0 \leqslant i \leqslant r} \max_{x \in \mathcal{R}} \left| f^{(n+1-i)}(x) \right|,\ \kappa_n(p) = \frac{(1 + \sqrt{1+4p})^n - (1 - \sqrt{1+4p})^n}{2^n \sqrt{1+4p}},$$
$$a^* = \max_{2 \leqslant i \leqslant n} |a_i|,\ \rho = a^* \operatorname{diam} \mathcal{R},$$

Theorem 3. *If partial numerators $a_i(x), i = \overline{2,n}$, of the finite functional continued fraction (FCF) of the form*

$$D_n(x) = \frac{P_n(x)}{Q_n(x)} = a_0(x) + \mathop{\mathrm{K}}_{i=1}^{n} \frac{a_i(x)}{1} \tag{8}$$

for arbitrary $x \in \mathcal{R}$ satisfy the condition of the Paydon–Wall type $|a_i(x)| \le t(1-t)$, where $0 < t \le \frac{1}{2}$, then the canonical denominator $Q_n(x)$ of the FCF (8) satisfies the inequality

$$|Q_n(x)| \ge \Omega_n(t) = \begin{cases} \dfrac{1 - \big(4(1-t)t\big)^{n+1}}{2^n(1 - 4(1-t)t)}, & \text{if} \quad 0 < t < \frac{1}{2}, \\ \dfrac{n+1}{2^n}, & \text{if} \quad t = \frac{1}{2}. \end{cases} \tag{9}$$

Proof. We rewrite the FCF (8) in the form of an equivalent continued fraction. In the formulas (4) we choose $r_i = 2, i = \overline{1,n}$. We get that

$$D_n(x) = \frac{P_n(x)}{Q_n(x)} = \frac{\bar{P}_n(x)}{\bar{Q}_n(x)} = \frac{a_0(x)}{2}\left(2 + \frac{4\frac{a_1(x)}{a_0(x)}}{2} + \frac{4a_2(x)}{2} + \cdots + \frac{4a_n(x)}{2} \right).$$

By the assumption of the theorem $|a_i(x)| \le (1-t)t, i = \overline{2,n}$, then $|\bar{Q}_1(x)| = 2 \ge 4(1-t)t + 1$. We use the Wallis formulas (3). We have

$$|\bar{Q}_2(x)| = |2\bar{Q}_1(x) + 4a_2(x)\bar{Q}_0 z)| \ge |2\bar{Q}_1(x)| - 4|a_2(x)| \ge |\bar{Q}_1(x)|(4(1-t)t + 1) - 4(1-t)t =$$

$$= |\bar{Q}_1(x)| + 4(1-t)t(|\bar{Q}_1(x)| - 1) \geq |\bar{Q}_1(x)| + \big(4(1-t)t\big)^2.$$

From this it follows that $|\bar{Q}_2(x)| - |\bar{Q}_1(x)| \geq \big(4(1-t)t\big)^2$. Next,

$$|\bar{Q}_3(x)| = |2\bar{Q}_2(x) + 4a_3(x)\bar{Q}_1(x)| \geq 2|\bar{Q}_2(x)| - 4|a_2(x)||\bar{Q}_1(x)| \geq |\bar{Q}_2(x)|\big(4(1-t)t+1\big) - \\ -4(1-t)t|\bar{Q}_1(x)| \geq |\bar{Q}_2(x)| + 4(1-t)t\big(|\bar{Q}_2(x)| - |\bar{Q}_1(x)|\big).$$

Then $|\bar{Q}_3(x)| - |\bar{Q}_2(x)| \geq \big(4(1-t)t\big)^3$. In the general case, for arbitrary $s = \overline{2,n}$ from the Wallis formulas, it follows

$$|\bar{Q}_s(x)| = |2\bar{Q}_{s-1}(x) + 4a_s(x)\bar{Q}_{s-2}(x)| \geq 2|\bar{Q}_{s-1}(x)| - 4|a_s(x)||\bar{Q}_{s-2}(x)| \geq \\ \geq |\bar{Q}_{s-1}(x)|\big(4(1-t)t+1\big) - 4(1-t)t|\bar{Q}_{s-2}(x)| \geq \\ \geq |\bar{Q}_{s-1}(x)| + 4(1-t)t\big(|\bar{Q}_{s-1}(x)| - |\bar{Q}_{s-2}(x)|\big).$$

Using induction we get that $|\bar{Q}_{s-1}(x)| - |\bar{Q}_{s-2}(x)| \geq \big(4(1-t)t\big)^s$. Then

$$|\bar{Q}_n(x)| = \sum_{i=2}^{n}\big(|\bar{Q}_i| - |\bar{Q}_{i-1}|\big) + |\bar{Q}_1| = \sum_{i=1}^{n}\big(4(1-t)t\big)^i = \begin{cases} \dfrac{1-\big(4(1-t)t\big)^{n+1}}{1-4(1-t)t}, & 0 < t < \frac{1}{2}, \\ n+1, & t = \frac{1}{2}. \end{cases}$$

Since $\bar{Q}_n(x) = 2^n Q_n(x)$ we get the estimate (9). □

The next statement follows from Theorems 2 and 3.

Theorem 4. *Let $f \in \mathbf{C}^{(n+1)}(\mathcal{R})$. Let C-ICF* (6) *be constructed by the values of the function f at nodes* (5) *and let the partial numerators of C-ICF satisfy the condition of the Paydon–Wall type, that is $0 < a^* \operatorname{diam}\mathcal{R} \leq p$. Then*

$$|f(x) - D_n(x)| \leq \frac{f^* \prod\limits_{k=0}^{n}|x - x_k|}{(n+1)!\,\Omega_n(t)}\Big(\kappa_{n+1}(p) + \sum_{m=1}^{r}\tbinom{n+1}{m}(a^*)^m\, m! \sum_{k=0}^{r-m}\tbinom{m+k}{m}\tbinom{n-m-k}{m+k}p^k\Big),$$

where

$$f^* = \max_{0\leq m\leq r}\max_{x\in\mathcal{R}}|f^{(n+1-m)}(x)|, \quad \kappa_n(p) = \frac{(1+\sqrt{1+4p})^n - (1-\sqrt{1+4p})^n}{2^n\sqrt{1+4p}},$$
$$a^* = \max_{2\leqslant i\leqslant n}|a_i|, \quad p = t(1-t), \; t \in (0;\tfrac{1}{2}], \; r = \big[\tfrac{n}{2}\big].$$

The main result of this paper is contained in the following theorem.

Theorem 5. *Let $f \in \mathbf{C}^{(n+1)}(\mathcal{R})$. Let C-ICF* (6) *be constructed by the values the function f at nodes* (5) *and let the partial numerators of C-ICF satisfy the condition of the Paydon–Wall type, that is $0 < a^* \operatorname{diam}\mathcal{R} \leq p$. Then*

$$|f(x) - D_n(x)| \leq \frac{f^* \prod\limits_{k=0}^{n}|x - x_k|}{(n+1)!\,\Omega_n(t)}\Big(\kappa_{n+1}(p) + \sum_{k=1}^{r}\tbinom{n+1}{k}(a^*)^k \sum_{i_1=1}^{n+1-2k}\kappa_{i_1}(p)\times \\ \times \sum_{i_2=i_1+2}^{n+3-3k}\kappa_{i_2-i_1-1}(p)\cdots\sum_{i_{k-1}=i_{k-2}+2}^{n-3}\kappa_{i_{k-1}-i_{k-2}-1}(p)\sum_{i_k=i_{k-1}+2}^{n-1}\kappa_{i_k-i_{k-1}-1}(p)\,\kappa_{n-i_k}(p)\Big), \quad (10)$$

where the quantities $r, p, a^*, f^*, \kappa_n(p)$ are defined in theorem 4.

3. Some properties of the continuant. Let $b_0, a_i, b_i, i \in \mathbb{N}$, be real numbers or functions. The determinant of the form

$$\mathcal{H}_n^{(i)} = \begin{vmatrix} b_i & a_{i+1} & 0 & 0 & \dots & 0 & 0 \\ -1 & b_{i+1} & a_{i+2} & 0 & \dots & 0 & 0 \\ 0 & -1 & b_{i+2} & a_{i+3} & \dots & 0 & 0 \\ 0 & 0 & -1 & b_{i+3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & b_{n-1} & a_n \\ 0 & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix}, \quad i = \overline{0,n}, \ n \in \mathbb{N},$$

is called a *continuant* ([10]). A continuant will be shortly written as

$$\mathcal{H}_n^{(i)} = \mathcal{K}\binom{a_{i+1}, a_{i+2}, \dots, a_{n-1}, a_n}{b_i, b_{i+1}, b_{i+2}, \dots, b_{n-1}, b_n}.$$

Since continuant is a partial case of the Hessenberg determinant, it satisfies the three-term recurrence relation ([11])

$$\mathcal{H}_m^{(i)} = b_m \mathcal{H}_{m-1}^{(i)} + a_m \mathcal{H}_{m-2}^{(i)}, \qquad m = \overline{i+1,n}, \quad H_i^{(i)} = b_i, \quad H_{i-1}^{(i)} = 1. \tag{11}$$

Theorem 6. *If the element a_k of a continuant is equal to zero, where $i < k \le n$, and all other elements are non-zero then*

$$\mathcal{H}_n^{(i)} = \mathcal{H}_n^{(k)} \cdot \mathcal{H}_{k-1}^{(i)}. \tag{12}$$

Proof. We have $a_k = 0$. From the recurrence relation (11) follows

$$\mathcal{H}_k^{(i)} = b_k \mathcal{H}_{k-1}^{(i)} = \mathcal{H}_k^{(k)} \mathcal{H}_{k-1}^{(i)}, \ \mathcal{H}_{k+1}^{(i)} = b_{k+1}\mathcal{H}_k^{(i)} + a_{k+1}\mathcal{H}_{k-1}^{(i)} = (b_k b_{k+1} + a_{k+1})\mathcal{H}_{k-1}^{(i)} = \mathcal{H}_{k+1}^{(k)}\mathcal{H}_{k-1}^{(i)}.$$

Therefore, for $n = k$ and $n = k+1$ the formula (12) holds. Let us assume that (12) holds for $n = m$. Then from (11) we get

$$\mathcal{H}_{m+1}^{(i)} = b_{m+1}\mathcal{H}_m^{(i)} + a_{m+1}\mathcal{H}_{m-1}^{(i)} = b_{m+1}\mathcal{H}_m^{(k)}\mathcal{H}_{k-1}^{(i)} + a_{m+1}\mathcal{H}_{m-1}^{(k)}\mathcal{H}_{k-1}^{(i)} = \mathcal{H}_{m+1}^{(k)}\mathcal{H}_{k-1}^{(i)}.$$

Thus, the formula (12) holds for arbitrary n. □

Theorem 7. *If $b_{k+i} = 0$, $b_s \ne 0$, when $s \ne k+i$, $a_s \ne 0, s = \overline{i,n}, i \le n$, then the following equality holds*

$$\mathcal{A}_n^{(i,k)} = \begin{vmatrix} b_i & a_{i+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & b_{i+1} & a_{i+2} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & b_{i+2} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{k+i-2} & a_{k+i-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & a_{k+i} & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & b_{k+i} & a_{k+i+1} & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & b_{k+i+1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & b_{n-1} & a_n \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix} =$$

$$= a_{k+i}\mathcal{K}_{k+i-2}^{(i)}\mathcal{K}_n^{(k+i+1)}, \ \textit{where} \quad \mathcal{K}_n^{(n+1)} = \mathcal{K}_0^{(i)} = 1, k = \overline{1, n-i}. \tag{13}$$

Proof. Let $k=1$. Then

$$\mathcal{A}_n^{(i,1)}=\begin{vmatrix} 0 & a_{i+1} & 0 & \dots & 0 & 0\\ -1 & b_{i+1} & a_{i+2} & \dots & 0 & 0\\ 0 & -1 & b_{i+2} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & b_{n-1} & a_n\\ 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix}.$$

We get $\mathcal{A}_n^{(i,1)}=a_{i+1}\mathcal{K}_{i-1}^{(i)}\mathcal{K}_n^{(i+2)}$, if we decompose the determinant consecutively by the 1st row and the 1st column. Let $k=2$. We decompose the determinant by the 2nd row and the 2nd column. We have $\mathcal{A}_n^{(i,2)}=a_{i+2}\mathcal{K}_i^{(i)}\mathcal{K}_n^{(i+3)}$. In the general case, for $k=m$ we will decompose the determinant consecutively by mth row and mth column, then

$$\mathcal{A}_n^{(i,m)}=\begin{vmatrix} b_i & a_{i+1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ -1 & b_{i+1} & a_{i+2} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ 0 & -1 & b_{i+2} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & b_{m+i-2} & a_{m+i-1} & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0 & a_{m+i} & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & -1 & b_{m+i} & a_{m+i+1} & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & b_{m+i+1} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & b_{n-1} & a_n\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix}=$$

$$=-a_{m+i}\begin{vmatrix} b_i & a_{i+1} & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0\\ -1 & b_{i+1} & a_{i+2} & \dots & 0 & 0 & 0 & \dots & 0 & 0\\ 0 & -1 & b_{i+2} & \dots & 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & b_{m+i-2} & a_{m+i-1} & 0 & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & -1 & a_{m+i+1} & \dots & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & 0 & b_{m+i+1} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & b_{n-1} & a_n\\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix}=$$

$$=-a_{m+i}a_{m+i-1}\begin{vmatrix} b_i & a_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 & 0\\ -1 & b_{i+1} & \dots & 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & b_{m+i-3} & a_{m+i-2} & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 0 & a_{m+i+1} & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 0 & b_{m+i+1} & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & b_{n-1} & a_n\\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix}+$$

$$+a_{m+i}\begin{vmatrix} b_i & a_{i+1} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & b_{i+1} & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{m+i-3} & a_{m+i-2} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & -1 & b_{m+i-2} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & b_{m+i+1} & a_{m+i+2} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 & b_{m+i+2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & b_{n-1} & a_n \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & b_n \end{vmatrix}.$$

We take the first $(m-2)$ columns in the determinants and use the Laplace theorem ([12]). We obtain $\mathcal{A}_n^{(i,m)} = a_{m+i}\mathcal{K}_{m+i-2}^{(i)}\mathcal{K}_n^{(m+i+1)}$. The formula (13) is proved. □

It is known ([10]) that a continuant has the property of invariance with respect to the inverse order of its elements, i.e.

$$\mathcal{K}\begin{pmatrix} a_{i+1}, a_{i+2}, \dots, a_{n-1}, a_n \\ b_i, b_{i+1}, b_{i+2}, \dots, b_{n-1}, b_n \end{pmatrix} = \mathcal{K}\begin{pmatrix} a_n, \ a_{n-1}, \dots, \ a_{i+3}, a_{i+2}, a_{i+1} \\ b_n, b_{n-1}, b_{n-2}, \dots, \ b_{i+2}, b_{i+1}, \ b_i \end{pmatrix}. \tag{14}$$

4. The representation C-ICF in the form of the ratio of the continuants. We use the fact that nth approximation (2) of a continued fraction (1) can be represented as a ratio of the continuants ([11]), i.e.

$$D_n = \frac{P_n}{Q_n} = \frac{\mathcal{H}_n^{(0)}}{\mathcal{H}_n^{(1)}}. \tag{15}$$

We introduce continuants of the form

$$\mathbf{C}_m^{(i)}(x) = \mathcal{K}\begin{pmatrix} a_{i+1}(x-x_i), a_{i+2}(x-x_{i+1}), \dots, a_m(x-x_{m-1}) \\ c_i, \quad 1, \quad 1, \quad \dots, \quad 1 \end{pmatrix}, \tag{16}$$

where $i = 0, 1$, $c_0 = a_0$, $c_1 = 1$, $m = \overline{1,n}$.

In accordance with (15) we have that C-ICF can be represented as the ratio of two continuants of the form (16), i.e. $D_n(x) = \mathbf{C}_n^{(0)}(x)/\mathbf{C}_n^{(1)}(x)$. Let us show that $D_n(x_k) = \mathbf{C}_n^{(0)}(x_k)/\mathbf{C}_n^{(1)}(x_k) = \mathbf{C}_k^{(0)}(x_k)/\mathbf{C}_k^{(1)}(x_k)$, $k = \overline{0,n}$. It is easy to see that the element $a_{k+i+1}(x-x_{k+i})$, $k = \overline{0,n}$, of the continuants $\mathbf{C}_n^{(i)}(x_k)$, $i = 0,1$, is equal to zero for $x = x_{k+i}$. By Theorem 6 we have

$$\frac{\mathbf{C}_n^{(0)}(x_k)}{\mathbf{C}_n^{(1)}(x_k)} = \frac{\mathbf{C}_k^{(0)}(x_k)\mathbf{C}_n^{(k+1)}(x_k)}{\mathbf{C}_n^{(1)}(x_k)\mathbf{C}_n^{(k+1)}(x_k)} = \frac{\mathbf{C}_k^{(0)}(x_k)}{\mathbf{C}_k^{(1)}(x_k)}.$$

We obtain another formula for determining the coefficients a_k, $k = \overline{0,n}$, of the C-ICF (6). The C-ICF satisfies the interpolation condition

$$y_k = D_n(x_k) = \frac{\mathcal{K}\begin{pmatrix} a_1(x_k-x_0), a_2(x_k-x_1), \dots, a_k(x_k-x_{k-1}) \\ a_0, \quad 1, \quad 1, \quad \dots, \quad 1 \end{pmatrix}}{\mathcal{K}\begin{pmatrix} a_2(x_k-x_1), a_3(x_k-x_2), \dots, a_k(x_k-x_{k-1}) \\ 1, \quad 1, \quad 1, \quad \dots, \quad 1 \end{pmatrix}},$$

or

$$
\begin{aligned}
&y_k\mathcal{K}\left(\begin{matrix} & a_2(x_k-x_1), a_3(x_k-x_2), \dots, a_k(x_k-x_{k-1}) \\ 1, & 1, \quad 1, \quad \dots, \quad 1 \end{matrix}\right) = \\
&= \mathcal{K}\left(\begin{matrix} & a_1(x_k-x_0), a_2(x_k-x_1), \dots, a_k(x_k-x_{k-1}) \\ a_0, & 1, \quad 1, \quad \dots, \quad 1 \end{matrix}\right).
\end{aligned}
$$

When it is considered that continuant has the property of invariance (14), then we have

$$
\begin{aligned}
&y_k\mathcal{K}\left(\begin{matrix} & a_k(x_k-x_{k-1}), a_{k-1}(x_k-x_{k-2}), \dots, a_2(x_k-x_1) \\ 1, & 1, \quad 1, \quad \dots, \quad 1 \end{matrix}\right) = \\
&= \mathcal{K}\left(\begin{matrix} & a_k(x_k-x_{k-1}), a_{k-1}(x_k-x_{k-2}), \dots, a_2(x_k-x_1), a_1(x_k-x_0) \\ 1, & 1, \quad 1, \quad \dots, \quad 1, \quad a_0 \end{matrix}\right).
\end{aligned}
$$

We decompose both continuants by the 1st rows. Then get it

$$
\begin{aligned}
&y_k\Bigg[\mathcal{K}\left(\begin{matrix} & a_{k-1}(x_k-x_{k-2}), a_{k-2}(x_k-x_{k-3}), \dots, a_2(x_k-x_1) \\ 1, & 1, \quad 1, \quad \dots, \quad 1 \end{matrix}\right) + \\
&+a_k(x_k-x_{k-1})\mathcal{K}\left(\begin{matrix} & a_{k-2}(x_k-x_{k-3}), a_{k-3}(x_k-x_{k-4}), \dots, a_2(x_k-x_1) \\ 1, & 1, \quad 1, \quad \dots, \quad 1 \end{matrix}\right)\Bigg] = \\
&= \mathcal{K}\left(\begin{matrix} & a_{k-1}(x_k-x_{k-2}), a_{k-2}(x_k-x_{k-3}), \dots, a_2(x_k-x_1), a_1(x_k-x_0) \\ 1, & 1, \quad 1, \quad \dots, \quad 1, \quad a_0 \end{matrix}\right) + \\
&+a_k(x_k-x_{k-1})\mathcal{K}\left(\begin{matrix} & a_{k-2}(x_k-x_{k-3}), a_{k-3}(x_k-x_{k-4}), \dots, a_2(x_k-x_1), a_1(x_k-x_0) \\ 1, & 1, \quad 1, \quad \dots, \quad 1, \quad a_0 \end{matrix}\right),
\end{aligned}
$$

whence

$$
\begin{aligned}
&-a_k(x_k-x_{k-1})\Bigg[\mathcal{K}\left(\begin{matrix} & a_{k-2}(x_k-x_{k-3}), \dots, a_2(x_k-x_1), a_1(x_k-x_0) \\ 1, & 1, \quad \dots, \quad 1, \quad a_0 \end{matrix}\right) - \\
&-y_k\mathcal{K}\left(\begin{matrix} & a_{k-2}(x_k-x_{k-3}), \dots, a_2(x_k-x_1), 0 \\ 1, & 1, \quad \dots, \quad 1, \quad 1 \end{matrix}\right)\Bigg] = \\
&= \mathcal{K}\left(\begin{matrix} & a_{k-1}(x_k-x_{k-2}), a_{k-2}(x_k-x_{k-3}), \dots, a_2(x_k-x_1), a_1(x_k-x_0) \\ 1, & 1, \quad 1, \quad \dots, \quad 1, \quad a_0 \end{matrix}\right) - \\
&-y_k\mathcal{K}\left(\begin{matrix} & a_{k-1}(x_k-x_{k-2}), a_{k-2}(x_k-x_{k-3}), \dots, a_2(x_k-x_1), 0 \\ 1, & 1, \quad 1, \quad \dots, \quad 1, \quad 1 \end{matrix}\right).
\end{aligned}
$$

Finally

$$
a_k = -\frac{\mathcal{K}\left(\begin{matrix} & a_{k-1}(x_k-x_{k-2}), \dots, a_2(x_k-x_1), a_1(x_k-x_0) \\ 1, & 1, \quad \dots, \quad 1, \quad y_k-a_0 \end{matrix}\right)}{(x_k-x_{k-1})\,\mathcal{K}\left(\begin{matrix} & a_{k-2}(x_k-x_{k-3}), \dots, a_2(x_k-x_1), a_1(x_k-x_0) \\ 1, & 1, \quad \dots, \quad 1, \quad y_k-a_0 \end{matrix}\right)}. \tag{17}
$$

If we take the common multiplier (-1) out from odd rows and even columns of a continuants of the numerator and the denominator (17) then we have

$$
a_k = -\frac{\mathcal{K}\left(\begin{matrix} & a_{k-1}(x_k-x_{k-2}), \dots, a_2(x_k-x_1), a_1(x_k-x_0) \\ -1, & -1, \quad \dots, \quad -1, \quad y_k-a_0 \end{matrix}\right)}{(x_k-x_{k-1})\,\mathcal{K}\left(\begin{matrix} & a_{k-2}(x_k-x_{k-3}), \dots, a_2(x_k-x_1), a_1(x_k-x_0) \\ -1, & -1, \quad \dots, \quad -1, \quad y_k-a_0 \end{matrix}\right)}. \tag{18}
$$

The formula (18) is equivalent to the formula (7).

5. Representation of remainder of C-ICF in the form of continuants. Let us express the remainder of C-ICF in terms of continuants. Consider the determinants $\left[\mathbf{c}_n^{(1)}(x)\right]_i^{(1)}$, $i=\overline{1,n}$, that are formed from the continuant $\mathbf{C}_n^{(1)}(x)$ by replacing the elements of the ith rows by their derivatives. Obviously, the determinant $\left[\mathbf{c}_n^{(1)}(x)\right]_i^{(1)}$ has only one non-zero element a_{i+1} in the ith row, $i=\overline{0,n-1}$ and $\left[\mathbf{c}_n^{(1)}(x)\right]_n^{(1)}\equiv 0$ because the last row of the determinant contains only zeros. Let $\left[\mathbf{c}_n^{(1)}(x)\right]_{ij}^{(2)}$, where $i,j=\overline{1,n}$, determinants whose elements of all rows, except the elements of ith and jth rows, equals the elements of the continuant $\mathbf{C}_n^{(1)}(x)$ and the elements of the ith and jth rows consist of the derivatives of elements of the corresponding rows.

The following identities are valid

$$\left[\mathbf{c}_n^{(1)}(x)\right]_{ii}^{(2)}\equiv\left[\mathbf{c}_n^{(1)}(x)\right]_{i,n}^{(2)}\equiv 0,\ i=\overline{1,n-2},\quad \left[\mathbf{c}_n^{(1)}(x)\right]_{i,i+1}^{(2)}\equiv 0,\ i=\overline{1,n-1}. \tag{19}$$

The first identity is obvious since the determinants contain a row with zero elements. The second identity can be obtained by decomposition the determinants by the Laplace rule in the sum of the products of the 2nd order minors on their cofactors contained in ith and $(i+1)$th rows.

Denote by $\left[\mathbf{c}_n^{(1)}(x)\right]_{i_1i_2\ldots i_k}^{(k)}$ determinants that are formed from the continuant $\mathbf{C}_n^{(1)}(x)$ by replacing the elements of the rows $i_1,i_2,\ldots,i_k$ by their derivatives.

Theorem 8. **(A)** *The derivative of the kth order, $k=\overline{1,[n/2]}$, of the continuant $\mathbf{C}_n^{(1)}(x)$ is equal to*

$$\left(\mathbf{C}_n^{(1)}(x)\right)^{(k)}=k!\sum_{i_1=1}^{n+1-2k}\sum_{i_2=i_1+2}^{n+3-2k}\cdots\sum_{i_k=i_{k-1}+2}^{n-1}\left[\mathbf{c}_n^{(1)}(x)\right]_{i_1i_2\ldots i_k}^{(k)}. \tag{20}$$

(B) *If $k>[n/2]$, then $\left(\mathbf{C}_n^{(1)}(x)\right)^{(k)}\equiv 0$.*

Proof. **(A)** We will prove the formula (20) by induction. By the rule of differentiation of the determinant [13] we have that $\left(\mathbf{C}_n^{(1)}(x)\right)^{(1)}=\sum_{i=1}^{n-1}\left[\mathbf{c}_n^{(1)}(x)\right]_i^{(1)}$, $\left[\mathbf{c}_n^{(1)}(x)\right]_n^{(1)}\equiv 0$. The 2nd derivative of the continuant $\mathbf{C}_n^{(1)}(x)$ will be equal to

$$\left(\mathbf{C}_n^{(1)}(x)\right)^{(2)}=\sum_{i_2=1}^{n}\sum_{i_1=1}^{n-1}\left[\mathbf{c}_n^{(1)}(x)\right]_{i_1i_2}^{(2)}.$$

Based on the formula (19) and the symmetry $\left[\mathbf{c}_n^{(1)}(x)\right]_{ij}^{(2)}=\left[\mathbf{c}_n^{(1)}(x)\right]_{ji}^{(2)}$, we have

$$\left(\mathbf{C}_n^{(1)}(x)\right)^{(2)}=2\cdot\sum_{i_1=1}^{n-3}\sum_{i_2=i_1+2}^{n-1}\left[\mathbf{c}_n^{(1)}(x)\right]_{i_1i_2}^{(2)}.$$

Suppose that the formula (20) holds for $k=m-1$, where $m-1<[n/2]$, i.e.

$$\left(\mathbf{C}_n^{(1)}(x)\right)^{(m-1)}=(m-1)!\sum_{i_1=1}^{n+3-2m}\sum_{i_2=i_1+2}^{n+5-2m}\cdots\sum_{i_{m-1}=i_{m-2}+2}^{n-1}\left[\mathbf{c}_n^{(1)}(x)\right]_{i_1i_2\ldots i_{m-1}}^{(m-1)}.$$

We find the derivative of the mth order of the continuant $\mathbf{C}_n^{(1)}(x)$. Then we have

$$\Big(\big(\mathbf{C}_n^{(1)}(x)\big)^{(m-1)}\Big)' = (m-1)! \sum_{i_m=1}^{n} \sum_{i_1=1}^{n+3-2m} \sum_{i_2=i_1+2}^{n+5-2m} \cdots \sum_{i_{m-1}=i_{m-2}+2}^{n-1} \big[\mathbf{c}_n^{(1)}(x)\big]^{(m)}_{i_1 i_2 \ldots i_{m-1} i_m}.$$

From (19) and the property $\big[\mathbf{c}_n^{(1)}(x)\big]^{(m)}_{i_1 i_2 \ldots i_m} = \big[\mathbf{c}_n^{(1)}(x)\big]^{(m1)}_{i_2 i_1 \ldots i_m} = \ldots = \big[\mathbf{c}_n^{(1)}(x)\big]^{(m)}_{i_m i_{m-1} \ldots i_1}$ follows

$$\big(\mathbf{C}_n^{(1)}(x)\big)^{(m)} = m! \sum_{i_1=1}^{n+1-2m} \sum_{i_2=i_1+2}^{n+3-2m} \cdots \sum_{i_m=i_{m-1}+2}^{n-1} \big[\mathbf{c}_n^{(1)}(x)\big]^{(m)}_{i_1 i_2 \ldots i_m}.$$

Therefore, the formula (20) is satisfied for an arbitrary $1 \le m \le [n/2]$.
(B) In accordance with **(A)**, the derivative of the $l = [n/2]$ order of the continuant $\mathbf{C}_n^{(1)}(x)$ is determined by the formula (20). Find the derivative of the $(l+1)$th order. We have

$$\Big(\mathbf{C}_n^{(1)}(x)\Big)^{(l+1)} = l! \sum_{k=1}^{n} \sum_{i_1=1}^{n+1-2l} \sum_{i_2=i_1+2}^{n+3-2l} \cdots \sum_{i_l=i_{l-1}+2}^{n-1} \big[\mathbf{c}_n^{(1)}(x)\big]^{(l+1)}_{i_1 i_2 \ldots i_m k}.$$

All the determinants on the right will be zero by to the formula (19). □

Theorem 9. *Let function $f \in \mathbf{C}^{(n+1)}(\mathcal{R})$ be interpolated on the set* (5) *by the C-ICF* (6). *Then there exists a point $\psi \in \operatorname{Int}\mathcal{R}$ such that*

$$f(x) - D_n(x) = \frac{\prod\limits_{i=0}^{n}(x-x_i)}{(n+1)!\,\mathbf{C}_n^{(1)}(x)} \Bigg(f^{(n+1)}(x)\mathbf{C}_n^{(1)}(x) + \sum_{k=1}^{r} \binom{n+1}{k} f^{(n+1-k)}(x) \times$$
$$\times \sum_{i_1=1}^{n+1-2k} \sum_{i_2=i_1+2}^{n+3-2k} \cdots \sum_{i_k=i_{k-1}+2}^{n-1} \big[\mathbf{c}_n^{(1)}(x)\big]^{(k)}_{i_1 i_2 \ldots i_k} \Bigg)\Bigg|_{x=\psi}, \quad r = [n/2]. \tag{21}$$

Proof. We have $f(x) - D_n(x) = f(x) - \mathbf{C}_n^{(0)}(x)/\mathbf{C}_n^{(1)}(x)$. Consider an auxiliary function

$$F(x) = f(x)\cdot \mathbf{C}_n^{(1)}(x) - \mathbf{C}_n^{(0)}(x) - \lambda(x-x_0)(x-x_1)\ldots(x-x_n). \tag{22}$$

The function F is zero in $(n+1)$ the interpolation nodes $x_i \in \mathcal{R}$, $i = \overline{0,n}$. If λ is taken as follows

$$\lambda = \frac{f(x_*)\cdot \mathbf{C}_n^{(1)}(x_*) - \mathbf{C}_n^{(0)}(x_*)}{(x_*-x_0)(x_*-x_1)\ldots(x_*-x_n)}, \qquad \text{where} \quad x_* \in \mathcal{R}\backslash X,$$

then the function F will be zero at $(n+2)$ points of the set $\tilde{X} = X \cup \{x_*\} \subset \mathcal{R}$. In accordance with the generalized Rolley theorem [14] there exist a point $\xi \in \mathbf{Int}\,\mathcal{R}$ such that $F^{(n+1)}(\xi) = 0$, or

$$\frac{d^{n+1}}{dx^{n+1}}\Big(f(x)\mathbf{C}_n^{(1)}(x)\Big)\Big|_{x=\xi} - \frac{d^{n+1}}{dx^{n+1}}\Big(\mathbf{C}_n^{(0)}(x)\Big)\Big|_{x=\xi} - (n+1)!\lambda = 0.$$

It follows from Theorem 8 that $\big(\mathbf{C}_n^{(1)}(x)\big)^{(n+1)} \equiv 0$. By the formula of the derivative of the $(n+1)$th order of the product of two functions, we have

$$\frac{d^{n+1}}{dx^{n+1}}\Big(f(x)\cdot \mathbf{C}_n^{(1)}(x)\Big) = \sum_{k=0}^{r} \binom{n+1}{k} f^{(n+1-k)}(x)\big(\mathbf{C}_n^{(1)}(x)\big)^{(k)}.$$

From this it follows

$$\lambda = \frac{1}{(n+1)!}\sum_{k=0}^{r}\left(\binom{n+1}{k} f^{(n+1-k)}(x)\sum_{i_1=1}^{n+1-2k}\sum_{i_2=i_1+2}^{n+3-2k}\cdots\sum_{i_k=i_{k-1}+2}^{n-1}\left[\mathbf{c}_n^{(1)}(x)\right]^{(k)}_{i_1i_2\ldots i_k}\right)\Big|_{x=\xi}.$$

Since x_* is an arbitrary point of the compact $\mathcal{R}$, dividing (22) by $\mathbf{C}_n^{(1)}(x)$ will have (21). □

6. Proof of Theorem 5. The main result of the article is proved in this paragraph.

Proof. The determinants $[\mathbf{c}_n^{(1)}(x)]^{(k)}_{i_1i_2\ldots i_k}$ from the formula (21) will be given through the continuants $\mathbf{C}_s^{(l)}(x)$. The ith row of the determinant $[\mathbf{c}_n^{(1)}(x)]^{(1)}_i$, $i=\overline{1,n-1}$, contains a single non-zero element a_{i+1}. Similarly, we get $[\mathbf{c}_n^{(1)}(x)]^{(2)}_{i_1i_2}=a_{i_1+1}a_{i_2+1}\mathbf{C}^{(1)}_{i_1-1}(x)\mathbf{C}^{(i_1+2)}_{i_2-1}(x)\mathbf{C}^{(i_2+2)}_{n}(x)$. By the induction can be shown formula

$$[\mathbf{c}_n^{(1)}(x)]^{(s)}_{i_1i_2\ldots i_s} = \prod_{k=1}^{s} a_{i_k+1}\cdot \mathbf{C}^{(1)}_{i_1-1}(x)\cdot \mathbf{C}^{(i_s+2)}_{n}(x)\cdot\prod_{k=2}^{s}\mathbf{C}^{(i_{k-1}+2)}_{i_k-1}(x), \qquad s=\overline{1,r}. \tag{23}$$

After substituting (23) in (21), we obtain

$$f(x)-D_n(x)=\frac{\prod\limits_{i=0}^{n}(x-x_i)}{(n+1)!\,\mathbf{C}_n^{(1)}(x)}\Big(f^{(n+1)}(x)\mathbf{C}_n^{(1)}(x)+\sum_{k=1}^{r}\binom{n+1}{k}f^{(n+1-k)}(x)\times$$
$$\times\sum_{i_1=1}^{n+1-2k}a_{i_1+1}\mathbf{C}^{(1)}_{i_1-1}(x)\sum_{i_2=i_1+2}^{n+3-2k}a_{i_2+1}\mathbf{C}^{(i_1+2)}_{i_2-1}(x)\cdots\sum_{i_{k-1}=i_{k-2}+2}^{n-3}a_{i_{k-1}+1}\mathbf{C}^{(i_{k-2}+2)}_{i_{k-1}-1}(x)\times$$
$$\times\sum_{i_k=i_{k-1}+2}^{n-1}a_{i_k+1}\mathbf{C}^{(i_{k-1}+2)}_{i_k-1}(x)\mathbf{C}^{(i_k+2)}_{n}(x)\Big)\Big|_{x=\psi},\qquad r=[n/2]. \tag{24}$$

In [6] the inequality $|\mathbf{C}_t^{(s)}(x)|\le\kappa_{t-s+2}(p)$ was proved. Since $|a_i|\le a^*, i=\overline{2,n}$, from (24) we get

$$|f(x)-D_n(x)|\le\frac{f^*\prod\limits_{k=0}^{n}|x-x_k|}{(n+1)!\,|\mathbf{C}_n^{(1)}(x)|}\Big(\kappa_{n+1}(p)+\sum_{k=1}^{r}\binom{n+1}{k}(a^*)^k\sum_{i_1=1}^{n+1-2k}\kappa_{i_1}(p)\times$$
$$\times\sum_{i_2=i_1+2}^{n+3-3k}\kappa_{i_2-i_1-1}(p)\cdots\sum_{i_{k-1}=i_{k-2}+2}^{n-3}\kappa_{i_{k-1}-i_{k-2}-1}(p)\sum_{i_k=i_{k-1}+2}^{n-1}\kappa_{i_k-i_{k-1}-1}(p)\,\kappa_{n-i_k}(p)\Big). \tag{25}$$

Coefficients of the C-ICF satisfy the conditions of the Paydon-Wall type. From (25) and Theorem 2 we obtain (10). □

The proven estimate of the remainder of the C-ICF has a complex form. We get an estimate of the remainder, which will be less accurate but more convenient.

From the definition of the $\kappa_s(p)$, it follows

$$\kappa_s(p)=\frac{(1+\sqrt{1+4p})^s-(1-\sqrt{1+4p})^s}{2^s\sqrt{1+4p}}\le\frac{(1+\sqrt{1+4p})^s}{2^{s-1}\sqrt{1+4p}},\quad s=\overline{1,n-1}.$$

Then

$$\sum_{i=s+2}^{n-1} \kappa_{i-s-1}(p)\,\kappa_{n-i}(p) \le \binom{n-s-2}{1} \frac{(1+\sqrt{1+4p})^{n-s-1}}{2^{n-s-3}(\sqrt{1+4p})^2}.$$

In the particular case for $s=-1$ we have

$$\sum_{i=1}^{n-1} \kappa_i(p)\,\kappa_{n-i}(p) \le \binom{n-1}{1} \frac{(1+\sqrt{1+4p})^{n}}{2^{n-2}(\sqrt{1+4p})^2}.$$

Similarly, we obtain

$$\sum_{i_1=s+2}^{n-3} \kappa_{i_1-s-1}(p) \sum_{i_2=i_1+2}^{n-1} \kappa_{i_2-i_1-1}(p)\,\kappa_{n-i_2}(p) \le \binom{n-s-3}{2} \frac{(1+\sqrt{1+4p})^{n-s-2}}{2^{n-s-5}(\sqrt{1+4p})^3}.$$

If $s=-1$ then

$$\sum_{i_1=1}^{n-3} \kappa_{i_1}(p) \sum_{i_2=i_1+2}^{n-1} \kappa_{i_2-i_1-1}(p)\,\kappa_{n-i_2}(p) \le \binom{n-2}{2} \frac{(1+\sqrt{1+4p})^{n-1}}{2^{n-4}(\sqrt{1+4p})^3}.$$

By induction, we prove that inequality holds for $m=\overline{1,r}$

$$\sum_{i_1=s+2}^{n+1-2m} \kappa_{i_1-s-1}(p) \sum_{i_2=i_1+2}^{n+3-2m} \kappa_{i_2-i_1-1}(p) \cdots \sum_{i_{m-1}=i_{m-2}+2}^{n-3} \kappa_{i_{m-1}-i_{m-2}-1}(p)\times$$
$$\times \sum_{i_m=i_{m-1}+2}^{n-1} \kappa_{i_m-i_{m-1}-1}(p)\,\kappa_{n-i_m}(p) \le \binom{n-s-m-1}{m} \frac{(1+\sqrt{1+4p})^{n-s-m}}{2^{n-s-2m-1}(\sqrt{1+4p})^{m+1}}.$$

If $m=1,2$, then the inequality holds. Suppose that it is true for $m=t$. Then for $m=t+1$ we have

$$\sum_{i_1=s+2}^{n-1-2t} \kappa_{i_1-s-1}(p) \cdots \sum_{i_t=i_{t-1}+2}^{n-3} \kappa_{i_t-i_{t-1}-1}(p) \sum_{i_{t+1}=i_t+2}^{n-1} \kappa_{i_{t+1}-i_t-1}(p)\,\kappa_{n-i_{t+1}}(p) \le$$
$$\le \sum_{i_1=s+2}^{n-1-2t} \binom{n-i_1-t-1}{t} \frac{(1+\sqrt{1+4p})^{n-i_1-t}}{2^{n-i_1-2t-1}(\sqrt{1+4p})^{t+1}} \frac{(1+\sqrt{1+4p})^{i_1-s-1}}{2^{i_1-s-2}\sqrt{1+4p}} =$$
$$= \binom{n-s-t-2}{t+1} \frac{(1+\sqrt{1+4p})^{n-s-t-1}}{2^{n-s-2t-3}(\sqrt{1+4p})^{t+2}}.$$

The inequality holds for $m=t+1$. In the particular case, for $s=-1$ we have

$$\sum_{i_1=1}^{n+1-2m} \kappa_{i_1}(p) \sum_{i_2=i_1+2}^{n+3-2m} \kappa_{i_2-i_1-1}(p) \cdots \sum_{i_{m-1}=i_{m-2}+2}^{n-3} \kappa_{i_{m-1}-i_{m-2}-1}(p)\times$$
$$\times \sum_{i_m=i_{m-1}+2}^{n-1} \kappa_{i_m-i_{m-1}-1}(p)\,\kappa_{n-i_m}(p) \le \binom{n-m}{m} \frac{(1+\sqrt{1+4p})^{n+1-m}}{2^{n-2m}\sqrt{1+4p})^{m+1}}, \quad m=\overline{1,r}. \tag{26}$$

If we take into account the (26) inequality, then (10) can be written

$$|f(x)-D_n(x)|\le \frac{f^*\prod\limits_{k=0}^{n}|x-x_k|}{(n+1)!\,\Omega_n(t)}\sum_{m=0}^{r}\binom{n+1}{m}\binom{n+1-m}{m}\frac{(a^*)^m(1+\sqrt{1+4p})^{n+1-m}}{2^{n-2m}(\sqrt{1+4p})^{m+1}}.$$

Let the value n be fixed. The sequence of binomial coefficients is unimodal, that is

$$\max_{0\le m\le r}\binom{n+1}{m}=\binom{n+1}{r}.$$

We then find the value of m, where $0\le m\le[\frac{n}{2}]$, for which the relations $\binom{n-m}{m}\ge\binom{n-m+1}{m-1}$, $\binom{n-m}{m}\ge\binom{n-m-1}{m+1}$ are valid. The first inequality holds when $(n-2m+2)(n-2m+1)\ge$ $\ge m(n-m+1)$. Square trinomial $5m^2-(5n+7)m+n^2+3n+2$ takes non-negative values when $m\in(0,(5n+7-\sqrt{5n^2+10n+9})/10)$ since $(5n+7+\sqrt{5n^2+10n+9})/10>[n/2]$. Similarly, the second inequality holds if $m\in((5n-3-\sqrt{5n^2+10n+9})/10),[\frac{n}{2}])$. Hence $\frac{5n-3-\sqrt{5n^2+10n+9}}{10}<m<\frac{5n+10-\sqrt{5n^2+10n+9}}{10}$. It follows

$$\max_{0\le m\le r}\binom{n-m}{m}=\binom{n-l}{l},\ l=[(5n+7-\sqrt{5n^2+10n+9})/10].$$

Next we have

$$\sum_{m=0}^{r}\frac{(a^*)^m\,(1+\sqrt{1+4p})^{n+1-m}}{2^{n-2m}\,(\sqrt{1+4p})^{m+1}}=\frac{(1+\sqrt{1+4p})^{n+2-r}}{2^n(\sqrt{1+4p})^r}\frac{(\sqrt{1+4p}(1+\sqrt{1+4p}))^r-(4a^*)^r}{\sqrt{1+4p}(1+\sqrt{1+4p})-4a^*}$$

as the sum of the geometric progression with the first term $(1+\sqrt{1+4p})^{n+1}/(2^n\sqrt{1+4p})$ and the denominator $q=(4a^*)/((1+\sqrt{1+4p})\sqrt{1+4p})$.

Finally we have an estimate of the remainder of the C-ICF

$$|f(x)-D_n(x)|\le\frac{f^*\prod\limits_{k=0}^{n}|x-x_k|}{(n+1)!\,\Omega_n(t)|}\binom{n+1}{r}\binom{n-l}{l}\frac{(1+\sqrt{1+4p})^{n+2-r}}{2^n(\sqrt{1+4p})^r}\times$$
$$\times\frac{(\sqrt{1+4p}(1+\sqrt{1+4p}))^r-(4a^*)^r}{\sqrt{1+4p}(1+\sqrt{1+4p})-4a^*}.$$

The obtained estimate of the remainder is simpler than the estimate of (10) but it is less accurate.

REFERENCES

1. I.P Gavrilyuk, V.L. Makarov, Calculation methods, Kyiv: Vysha Shcola, 1995, 367 p. (in Ukrainian)
2. A.A. Privalov, Theory of the interpolation of functions, Saratov: Izdatel'stvo Saratovskogo Universiteta, 1990, 424 p. (in Russian)
3. V.K. Dzyadyk, I.A. Shevchuk, Theory of uniform approximation of functions by polynomials. Walter de Gruyter, Berlin–New York, 2008.
4. V.L. Makarov, V.V. Khlobystov, Spline approximation of functions, Moskva: Vyshaya Shcola, 1983, 80 p. (in Russian)

5. G.A. Baker, P. Graves–Morris, Padé Approximants, Addison–Wesley Publ. Comp., 1981, 540 p.
6. M.M. Pahirya, Approximation of functions by continued fractions, Uzhhorod: Grazda, 2016, 412 p. (in Ukrainian)
7. M.M. Pahirya, *Estimation of the remainder for the interpolation continued C-fraction*, Ukr. Mat. Zh., **66** (2014), №6, 806–814, (in Ukrainian); Engl. trasl. Ukr. Math. J., **66** (2014), №6, 905–915.
8. W.B. Jones, W.J. Thron, Continued fractions. Analytic theory and applications, Addison–Wesley Publ. Comp., 1980, 428 p.
9. T.N. Thiele, Interpolationsprechnung, Leipzig: Commisission von B.G. Teubner, 1909, XII+175S.
10. G. Chrystal, Algebra, Vol. II, London, A.C. Black, 1889, XXIV+616 p.
11. R. Vein, P. Dale, Determinants and their application in mathematical physics, Springer Science & Business Media, 2006, VIII+376 p.
12. A.G. Kurosh, A course in higher algebra, Moscow.: Mir, 1988, 432 p. (in Russian)
13. G.M. Fihtengol'c, A course in differential and integral calculus. V.I., Moskva, Nauka, 2003, 680 p. (in Russian)
14. K.I. Babenko, Foundations of numerical analysis, Moskva–Izhevsk: Regularnaya & Haoticheskaya dynamika, 2002, 848 p. (in Russian)

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