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PROBLEM OF INTERPOLATION OF FUNCTIONS BY TWO-DIMENSIONAL CONTINUED FRACTIONS

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We investigate the problem of interpolation of functions of two real variables by two-dimensional continued fractions.

1. Introduction

The problem of interpolation of functions of two real variables by two-dimensional continued fractions was studied by Kuchmins'ka [1] and Cuyt [2]. Later, this problem was investigated in [3–6], where, in particular, a somewhat different algorithm for the determination of the coefficients of the interpolational two-dimensional continued fraction was proposed and the method was generalized to the problem of interpolation of functions of three real variables by three-dimensional continued fractions. Other types of interpolational two-dimensional continued fractions were considered in [7–9]. In the present paper, we continue the investigations begun in [9].

2. Interpolational Two-Dimensional Continued Fractions

Consider the two-dimensional continued fraction

$$D(x,y) = \Phi_0(x,y) + \mathop{\mathrm{K}}_{i=1}^{\infty} \frac{a_{ii}(x,y)}{\Phi_i(x,y)}, \tag{1}$$

where

$$\Phi_i(x,y) = b_{ii}(x,y) + \mathop{\mathrm{K}}_{j=i+1}^{\infty} \frac{a_{ji}(x,y)}{b_{ji}(x,y)} + \mathop{\mathrm{K}}_{j=i+1}^{\infty} \frac{a_{ij}(x,y)}{b_{ij}(x,y)}, \quad i = 0, 1, \ldots,$$

$a_{ij}(x,y) \not\equiv 0,$ and $b_{ij}(x,y)$ are functions of two variables.

Definition 1. *The finite functional two-dimensional continued fraction*

$$D_{(n_x,n_y)}(x,y) = \Phi_0^{(n_x,n_y)}(x,y) + \mathop{\mathrm{K}}_{i=1}^{n} \frac{a_{ii}(x,y)}{\Phi_i^{(n_x,n_y)}(x,y)}, \quad n = \min\{n_x, n_y\}, \tag{2}$$

where

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$$\Phi_i^{(n_x,n_y)}(x,y) = b_{ii}(x,y) + \mathop{\mathrm{K}}_{j=i+1}^{n_x} \frac{a_{ji}(x,y)}{b_{ji}(x,y)} + \mathop{\mathrm{K}}_{j=i+1}^{n_y} \frac{a_{ij}(x,y)}{b_{ij}(x,y)}, \quad i = 0, 1, \ldots, n,$$

is called the (n_x, n_y)*-convergent of the two-dimensional continued fraction (1).*

In what follows, we assume that

$$\mathop{\mathrm{K}}_{s=r}^{t} \frac{a_s}{b_s} = 0 \quad \text{if} \quad r > t.$$

Denote $n_{xy} = (n_x, n_y)$.

Using an analog of the inverse recurrence algorithm [4, 5], we represent the two-dimensional continued fraction (2) in the form $D_{n_{xy}}(x,y) = \dfrac{P_{n_{xy}}(x,y)}{Q_{n_{xy}}(x,y)}$, where $P_{n_{xy}}(x,y)$ is the numerator and $Q_{n_{xy}}(x,y)$ is the denominator of convergent (2).

3. Relation for the Difference of Convergents

Using the methods of [10], we can find a relation for the difference of convergents. Denote the remainder [the tail of the two-dimensional continued fraction (2)] by

$$Q_k^{n_{xy}} = \Phi_k^{n_{xy}}(x,y) + \mathop{\mathrm{K}}_{i=k+1}^{n} \frac{a_{ii}(x,y)}{\Phi_i^{n_{xy}}(x,y)}, \quad k = 0, \ldots, n-1,$$
$$Q_n^{n_{xy}} = \Phi_n^{n_{xy}}(x,y). \tag{3}$$

Let $n_{xy} + 1 = (n_x + 1, n_y + 1)$. Then

$$D_{n_{xy}+1} - D_{n_{xy}} = \sum_{m=0}^{n} \left(\frac{(-1)^{n_x+1} \prod_{j=m+1}^{n_x+1} a_{jm}}{Q_{n_x,m}\, Q_{n_x+1,m}} + \frac{(-1)^{n_y+1} \prod_{j=m+1}^{n_y+1} a_{mj}}{Q_{m,n_y}\, Q_{m,n_y+1}} \right) \prod_{s=1}^{m} \frac{a_{ss}}{Q_s^{n_{xy}}\, Q_s^{n_{xy}+1}}$$
$$+ \frac{(-1)^n \prod_{s=1}^{n+1} a_{ss}}{Q_{n+1}^{n_{xy}+1} \prod_{s=1}^{n} Q_s^{n_{xy}}\, Q_s^{n_{xy}+1}}, \tag{4}$$

where $Q_{n_x,m}$ is the denominator of the continued fraction $\mathop{\mathrm{K}}_{i=m+1}^{n_x} \dfrac{a_{im}}{b_{im}}$ and Q_{m,n_y} is the denominator of the continued fraction $\mathop{\mathrm{K}}_{i=m+1}^{n_y} \dfrac{a_{mi}}{b_{mi}}$ $\big(Q_{n_x+1,m}$ and Q_{m,n_y+1} are defined by analogy$\big)$.

4. Relation for the Remainder of an Interpolational Two-Dimensional Continued Fraction

Let a function of two variables $f(x,y)$ be continuous together with its partial derivatives up to the (k_x+1)th order with respect to x and up to the (k_y+1)th order with respect to y on the set $G = [\alpha_x, \beta_x] \times [\alpha_y, \beta_y]$.

We choose the decompositions $X = \{x_i\colon x_i \in [\alpha_x, \beta_x],\ x_i \neq x_l \text{ for } i \neq l,\ i, l = 0, 1, \ldots, k_x\}$ and $Y = \{y_j\colon y_j \in [\alpha_y, \beta_y],\ y_j \neq y_l \text{ for } j \neq l,\ j, l = 0, 1, \ldots, k_y\}$. The Cartesian product of these sets $G_{k_{xy}} = X \times Y = \{(x_i, y_j)\colon x_i \in X,\ y_j \in Y\}$ forms a grid in the set G. Assume that we have the functional two-dimensional continued fraction (2), where $n_x = n_x(k_x)$ and $n_y = n_y(k_y)$.

Definition 2. *The finite functional two-dimensional continued fraction (2) is called an interpolational two-dimensional continued fraction if the following relations hold at the grid nodes* $G_{k_{xy}}$:

$$D_{n_{xy}}(x_i, y_j) = c_{ij}, \tag{5}$$

where $c_{ij} = f(x_i, y_j),\ i = 0, 1, \ldots, k_x,\ j = 0, 1, \ldots, k_y$.

If the two-dimensional continued fraction (2) is an interpolational one, then the difference

$$R_{n_{xy}}(x, y) = f(x, y) - \frac{P_{n_{xy}}(x, y)}{Q_{n_{xy}}(x, y)}$$

is called the remainder of the interpolational two-dimensional continued fraction. Assume that the partial numerators $a_{ij}(x, y)$ and partial denominators $b_{ij}(x, y)$ are polynomials. Using Theorem 1 from [11], one can prove the following statement:

Theorem 1 [9]. *Suppose that* $f(x, y) \in \mathbf{C}^{(k_x+1, k_y+1)}(G)$, *the two-dimensional continued fraction (2) is an interpolational one, the numerator* $P_{n_{xy}}(x, y)$ *and denominator* $Q_{n_{xy}}(x, y)$ *of fraction (2) are polynomials,* $\deg_x P_{n_{xy}}(x, y) \leq k_x$, *and* $\deg_y P_{n_{xy}}(x, y) \leq k_y$. *Then there exist* $\xi, \theta \in (\alpha_x, \beta_x)$ *and* $\eta, \nu \in (\alpha_y, \beta_y)$ *such that*

$$\begin{aligned} R_{n_{xy}}(x, y) &= f(x, y) - \frac{P_{n_{xy}}(x, y)}{Q_{n_{xy}}(x, y)} \\ &= \frac{1}{Q_{n_{xy}}(x, y)} \left[\frac{\omega_{k_x}(x)}{(k_x+1)!} \left. \frac{\partial^{k_x+1} h(x, y)}{\partial x^{k_x+1}} \right|_{x=\theta} + \frac{\omega_{k_y}(y)}{(k_y+1)!} \left. \frac{\partial^{k_y+1} h(x, y)}{\partial y^{k_y+1}} \right|_{y=\nu} \right. \\ &\qquad \left. + \frac{\omega_{k_x}(x)\, \omega_{k_y}(y)}{(k_x+1)!\, (k_y+1)!} \left. \frac{\partial^{k_x+k_y+2} h(x, y)}{\partial x^{k_x+1} \partial y^{k_y+1}} \right|_{\substack{x=\xi \\ y=\eta}} \right], \end{aligned} \tag{6}$$

where

$$\omega_{k_x}(x) = \prod_{i=0}^{k_x} (x - x_i), \quad \omega_{k_y}(y) = \prod_{j=0}^{k_y} (y - y_j), \quad h(x, y) = Q_{n_{xy}}(x, y)\, f(x, y).$$

5. Kuchmins'ka–Cuyt-Type Interpolational Two-Dimensional Continued Fractions

Consider several types of interpolational two-dimensional continued fractions. We begin with an interpolational two-dimensional continued fraction proposed by Kuchmins'ka [1] and Cuyt [2]. Assume that the partial numerators $a_{ij}(x, y)$ in the interpolational two-dimensional continued fraction (2) are defined by the formula

$$a_{ij}(x,y) = \begin{cases} x - x_{i-1} & \text{for} \quad i > j, \\ y - y_{j-1} & \text{for} \quad i < j, \\ (x - x_{i-1})(y - y_{i-1}) & \text{for} \quad i = j, \end{cases}$$

the denominators b_{ij} are the coefficients, $n_x = k_x$, and $n_y = k_y$. In this case, we have the Kuchmins'ka–Cuyt interpolational two-dimensional continued fraction

$$D_{n_{xy}}(x,y) = \frac{P_{n_{xy}}(x,y)}{Q_{n_{xy}}(x,y)} = \Phi_0^{n_{xy}}(x,y) + \mathop{\mathrm{K}}_{k=1}^{n} \frac{(x - x_{k-1})(y - y_{k-1})}{\Phi_k^{n_{xy}}(x,y)},$$
$$n = \min\{n_x, n_y\}, \tag{7}$$

where

$$\Phi_k^{n_{xy}}(x,y) = b_{kk} + \mathop{\mathrm{K}}_{i=k+1}^{n_x} \frac{x - x_{i-1}}{b_{ik}} + \mathop{\mathrm{K}}_{i=k+1}^{n_y} \frac{y - y_{i-1}}{b_{ki}}.$$

Theorem 2 [5]. *The interpolational two-dimensional continued fraction (7) is a fractional rational function of two independent variables. The degrees of the polynomials $P_{n_{xy}}(x,y)$ and $Q_{n_{xy}}(x,y)$ in x and y satisfy the inequalities $\deg_k P_{n_{xy}}(x,y) \le r(n_k)$ and $\deg_k Q_{n_{xy}}(x,y) \le r(n_k) + \varepsilon(n_k)$, where*

$$r(n_k) = \frac{(n_k+1)^2 + \varepsilon(n_k+1)}{4} \qquad \textit{and} \qquad \varepsilon(n_k) = \frac{(-1)^{n_k} - 1}{2}, \quad k \in \{x, y\}.$$

It is easy to see that the number of the coefficients of the interpolational two-dimensional continued fraction (7) is equal to the number of the interpolation nodes in $G_{n_{xy}}$. The coefficients of the interpolational two-dimensional continued fraction (7) can be determined by the Kuchmins'ka–Cuyt algorithm of inverse divided differences [1, 2] or directly from condition (5). Consider the matrices

$$\mathbf{X} = (x_{ij})_{i,j=0,1,\ldots,n_x}, \quad x_{ij} = \begin{cases} x_i - x_j & \text{for} \quad i > j, \\ 1 & \text{for} \quad i \le j, \end{cases} \tag{8}$$

and

$$\mathbf{Y} = (y_{ij})_{i,j=0,1,\ldots,n_y}, \quad y_{ij} = \begin{cases} y_i - y_j & \text{for} \quad i > j, \\ 1 & \text{for} \quad i \le j. \end{cases} \tag{9}$$

For functions of two variables, the partial inverse divided difference of the kth order is defined by the relation

$$\delta_{ij}^k = \frac{x_{ik} y_{jk}}{\delta_{ij}^{k-1} + \theta_j^k \delta_{ik}^{k-1} + \theta_i^k \delta_{kj}^{k-1} + \theta_i^k \theta_j^k \delta_{kk}^{k-1}}, \qquad \delta_{ij}^{-1} = c_{ij},$$

$$\theta_s^t = \begin{cases} -1 & \text{for} \quad s > t, \\ 0 & \text{for} \quad s \le t, \end{cases}$$

$$i = 0, 1, \ldots, n_x, \quad j = 0, 1, \ldots, n_y, \quad k = 0, \ldots, N-1,$$

$$N = \max\{n_x, n_y\}, \quad i, j > k.$$

Proposition 1 [5, 6]. *The coefficients of the interpolational two-dimensional continued fraction (7) satisfy the relation*

$$b_{ij} = \delta_{ij}^{s-1}, \tag{10}$$

where $i = 0, 1, \ldots, n_x,$ $j = 0, 1, \ldots, n_y,$ *and* $s = \max\{i, j\}.$

6. Estimate for the Remainder of the Kuchmins'ka–Cuyt Interpolational Two-Dimensional Continued Fraction

We use the following statement for continued fractions:

Theorem 3 [12]. *If all partial numerators* a_k *and partial denominators* b_k *of the continued fraction*

$$\frac{P_m^{(n)}}{Q_m^{(n)}} = \mathop{\mathrm{K}}_{k=m}^{n} \frac{a_k}{b_k}$$

satisfy the conditions $|a_k| \le d$ *and* $|b_k| \ge d + 1,$ *then*

$$|Q_m^{(n)}| \ge \begin{cases} \dfrac{d^{n+1-m} - 1}{d - 1} & \textit{for} \quad d \ne 1, \\ n + 1 - m & \textit{for} \quad d = 1. \end{cases}$$

Theorem 4. *Suppose that the following conditions are satisfied:*

(i) for a function $f(x, y)$ *continuous in the domain* $G,$ *the interpolational two-dimensional continued fraction (7) is defined, the coefficients of which are determined by the values of the function at the grid nodes* $G_{n_{xy}}$ *according to formulas (10);*

(ii) the coefficients of the interpolational two-dimensional continued fraction (7) satisfy the conditions $|b_{ij}| \ge d_x + 1,$ $|b_{ji}| \ge d_y + 1,$ $i > j,$ *and* $|b_{ii}| \ge d_x d_y + 3,$ $i = 1, \ldots, n,$ *where* $d_x = \beta_x - \alpha_x$ *and* $d_y = \beta_y - \alpha_y;$

(iii) there exists a point $(x_*, y_*) \in G,$ $x_* \notin X,$ $y_* \notin Y,$ *such that* $|b_{n_x+1,j}(x_*)| \ge d_x + 1,$ $|b_{i,n_y+1}(y_*)| \ge d_y + 1,$ $i = 0, 1, \ldots, n_x,$ $j = 0, \ldots, n_y,$ *and* $|b_{n+1,n+1}(x_*, y_*)| \ge d_x\, d_y + 3,$ *where the coefficients* $b_{n_x+1,j}(x_*),$ $b_{i,n_y+1}(y_*),$ *and* $b_{n+1,n+1}(x_*, y_*)$ *are determined by relations (10) with* $x_{n_x+1} = x_*$ *and* $y_{n_y+1} = y_*.$

Then the following inequality is true:

$$\left|f(x_*, y_*) - D_{n_{xy}}(x_*, y_*)\right|$$

$$\leq \begin{cases} \sum_{m=0}^{n} \dfrac{d_x^{n_x+1}(d_x-1)^2}{d_y^m(d_x^{n_x+1}-d_x^m)(d_x^{n_x+2}-d_x^m)} + \sum_{m=0}^{n} \dfrac{d_y^{n_y+1}(d_y-1)^2}{d_x^m(d_y^{n_y+1}-d_y^m)(d_y^{n_y+2}-d_y^m)} + \dfrac{1}{d_x^n d_y^n}, \\ \qquad d_x \neq 1, \quad d_y \neq 1, \\ \sum_{m=0}^{n} \dfrac{1}{d_y^m(n_x+1-m)(n_x+2-m)} + \sum_{m=0}^{n} \dfrac{d_y^{n_y+1}(d_y-1)^2}{(d_y^{n_y+1}-d_y^m)(d_y^{n_y+2}-d_y^m)} + \dfrac{1}{d_y^n}, \\ \qquad d_x = 1, \quad d_y \neq 1, \\ \sum_{m=0}^{n} \dfrac{d_x^{n_x+1}(d_x-1)^2}{(d_x^{n_x+1}-d_x^m)(d_x^{n_x+2}-d_x^m)} + \sum_{m=0}^{n} \dfrac{1}{d_x^m(n_y+1-m)(n_y+2-m)} + \dfrac{1}{d_x^n}, \\ \qquad d_x \neq 1, \quad d_y = 1, \\ \sum_{m=0}^{n} \dfrac{1}{(n_x+1-m)(n_x+2-m)} + \sum_{m=0}^{n} \dfrac{1}{(n_y+1-m)(n_y+2-m)} + 1, \\ \qquad d_x = 1, \quad d_y = 1. \end{cases}$$

Proof. Let us choose the point (x_*, y_*). By virtue of the conditions of the theorem, we have $x_* \notin X$ and $y_* \notin Y$. Using the values of the function $f(x,y)$ at the grid nodes $G_{n_{xy}+1} = \{x_0, \ldots, x_{n_x}, x_{n_x+1}\} \times \{y_0, \ldots, y_{n_y}, y_{n_y+1}\}$, where $x_{n_x+1} = x_*$ and $y_{n_y+1} = y_*$, we construct one more interpolational two-dimensional continued fraction as follows:

$$D_{n_{xy}+1}(x,y) = \Phi_0^{n_{xy}+1}(x,y) + \mathop{\mathrm{K}}_{k=1}^{n+1} \frac{(x-x_{k-1})(y-y_{k-1})}{\Phi_k^{n_{xy}+1}(x,y)}, \tag{11}$$

where

$$\Phi_k^{n_{xy}+1}(x,y) = b_{kk} + \mathop{\mathrm{K}}_{i=k+1}^{n_x+1} \frac{x-x_{i-1}}{b_{ik}} + \mathop{\mathrm{K}}_{i=k+1}^{n_y+1} \frac{y-y_{i-1}}{b_{ki}}, \quad k = 0, 1, \ldots, n+1.$$

It is easy to see that the coefficients $b_{ij},\ i = 0, 1, \ldots, n_x,\ j = 0, 1, \ldots, n_y,$ in the interpolational two-dimensional continued fraction (11) are equal to the corresponding coefficients in the interpolational two-dimensional continued fraction (7) by construction, and the coefficients $b_{n_x+1,i} = b_{n_x+1,i}(x_*),\ b_{i,n_y+1} = b_{i,n_y+1}(y_*),$ and $b_{n+1,n+1} = b_{n+1,n+1}(x_*, y_*)$ are determined by relations (10).

The continued fraction (11) is an interpolational one, i.e., $D_{n_{xy}+1}(x_*, y_*) = f(x_*, y_*)$ by construction. We have

$$f(x_*, y_*) - D_{n_{xy}}(x_*, y_*) = D_{n_{xy}+1}(x_*, y_*) - D_{n_{xy}}(x_*, y_*). \tag{12}$$

The difference of the convergents $D_{n_{xy}+1}(x_*, y_*) - D_{n_{xy}}(x_*, y_*)$ is determined by relation (4) for $a_{jm} = x - x_{j-1}$, $j = m+1, \ldots, n_x+1$, $a_{mj} = y - y_{j-1}$, $j = m+1, \ldots, n_y+1$, $m = 0, 1, \ldots, n$, and $a_{ss} = (x - x_{s-1})(y - y_{s-1})$, $s = 1, 2, \ldots, n$.

Using the method of complete mathematical induction, we prove that

$$\left|Q_k^{n_{xy}}\right| \geq d_x d_y, \quad k = 1, 2, \ldots, n, \qquad \left|Q_k^{n_{xy}+1}\right| \geq d_x d_y, \quad k = 1, 2, \ldots, n+1. \tag{13}$$

The moduli of the denominators $Q_{n_x,m}$, $Q_{n_x+1,m}$, Q_{m,n_y}, and Q_{m,n_y+1} of the continued fractions are estimated according to Theorem 3, which completes the proof of the theorem.

7. Interpolational Two-Dimensional Continued C′-Fraction

Consider an interpolational two-dimensional continued fraction in the form of the C′-fraction

$$D_{n_{xy}}(x, y) = b_{00} + \Phi_0^{n_{xy}}(x, y) + \mathop{\mathrm{K}}_{i=1}^{n} \frac{b_{ii}(x - x_{i-1})(y - y_{i-1})}{1 + \Phi_i^{n_{xy}}(x, y)}, \quad n = \min\{n_x, n_y\}, \tag{14}$$

where

$$\Phi_i^{n_{xy}}(x, y) = \mathop{\mathrm{K}}_{j=i+1}^{n_x} \frac{b_{ji}(x - x_{j-1})}{1} + \mathop{\mathrm{K}}_{j=i+1}^{n_y} \frac{b_{ij}(y - y_{j-1})}{1}, \quad i = 0, 1, \ldots, n.$$

We define the coefficients of the interpolational two-dimensional continued C′-fraction (14) so that condition (5) is satisfied at the nodes of the set $G_{n_{xy}}$. Denote

$$\beta_{ij}^{(k)} = \frac{\omega_{ij}^{(k-1)}}{x_{ik} y_{j\,k}} \left[\frac{1}{\beta_{ij}^{(k-1)}} + \frac{\theta_j^k}{\beta_{ik}^{(k-1)}} + \frac{\theta_i^k}{\beta_{kj}^{(k-1)}} + \frac{\theta_j^k\,\theta_i^k}{\beta_{kk}^{(k-1)}}\right], \tag{15}$$

where

$$\omega_{ij}^{(k-1)} = \begin{cases} \beta_{ik}^{(k-1)} & \text{for} \quad j > i, \quad i < k, \\ \beta_{kj}^{(k-1)} & \text{for} \quad i > j, \quad j < k, \\ \beta_{kk}^{(k-1)} & \text{for} \quad i \geq k, \quad j \geq k, \end{cases}$$

$$\beta_{ij}^{(0)} = \frac{c_{ij} + \theta_j^0 c_{i0} + \theta_i^0 c_{0j} + \theta_j^0 \theta_i^0 c_{00}}{x_{i0}\, y_{j0}},$$

$$i = 0, 1, \ldots, n_x, \quad j = 0, 1, \ldots, n_y, \quad k = 1, 2, \ldots, N-1, \quad N = \max\{n_x, n_y\}.$$

Proposition 2. *The coefficients of the interpolational two-dimensional continued fraction (14) can be determined by the relation*

$$b_{ij} = \beta_{ij}^{(k-1)}, \qquad i = 0, 1, \ldots, n_x, \quad j = 0, 1, \ldots, n_y, \quad k = \max\{i, j\}. \tag{16}$$

Proof. We prove formula (16) by the method of complete mathematical induction by analogy with [4]. It is easy to see that this formula holds for the coefficients $\Phi_0^{n_{xy}}(x, y)$ [9] for any n_x and n_y. For $k = 0, \ldots, n_x$ and $m = 0, \ldots, n_y$, the following equality is true:

$$\Phi_0^{n_{xy}}(x_k, y_m) = \mathop{\mathrm{K}}_{j=1}^{n_x} \frac{b_{j0} x_{kj-1}}{1} + \mathop{\mathrm{K}}_{j=1}^{n_y} \frac{b_{0j} y_{mj-1}}{1} = c_{k0} + c_{0m} - 2\, b_{00}. \tag{17}$$

Assume that the coefficients $\Phi_k^{n_{xy}}(x, y)$, $k = 1, 2, \ldots, n$, are determined by (16) for $n = t - 1$. Let $n = t$. We have

$$D_{t_{xy}}(x, y) = b_{00} + \Phi_0^{t_{xy}}(x, y) + \cfrac{b_{11}\,(x - x_0)\,(y - y_0)}{1 + \Phi_1^{t_{xy}}(x, y) + \mathop{\mathrm{K}}\limits_{i=2}^{t} \cfrac{b_{ii}\,(x - x_{i-1})\,(y - y_{i-1})}{1 + \Phi_i^{t_{xy}}(x, y)}}. \tag{18}$$

Denote

$$\mu(x, y) = 1 + \Phi_1^{t_{xy}}(x, y) + \mathop{\mathrm{K}}_{i=2}^{t} \frac{b_{ii}\,(x - x_{i-1})\,(y - y_{i-1})}{1 + \Phi_i^{t_{xy}}(x, y)}. \tag{19}$$

Then we rewrite (18) in the form

$$D_{t_{xy}}(x, y) = b_{00} + \Phi_0^{t_{xy}}(x, y) + \frac{b_{11}\,(x - x_0)\,(y - y_0)}{\mu(x, y)}.$$

Since $D_{t_{xy}}(x_i, y_j) = c_{ij}$ for $i = 0, 1, \ldots, t_x$ and $j = 0, 1, \ldots, t_y$, taking (17) into account we get

$$\mu_{ij} = \mu(x_i, y_j) = \frac{b_{11} x_{i0} y_{j0}}{c_{ij} - c_{i0} - c_{0j} - c_{00}}.$$

The two-dimensional continued fraction (19) has $t - 1$ levels, and its coefficients are, by assumption, determined by relation (16). Thus, we have

$$b_{ij} = \widetilde{\beta}_{ij}^{(k-1)}, \qquad i = 1, 2, \ldots, t_x, \quad j = 1, 2, \ldots, t_y, \quad k = \max\{i, j\}, \tag{20}$$

where

$$\widetilde{\beta}_{ij}^{(k)} = \frac{\widetilde{\omega}_{ij}^{(k-1)}}{x_{ik} y_{jk}} \left[\frac{\theta_j^k\, \theta_i^k}{\widetilde{\beta}_{kk}^{(k-1)}} + \frac{\theta_j^k}{\widetilde{\beta}_{ik}^{(k-1)}} + \frac{\theta_i^k}{\widetilde{\beta}_{kj}^{(k-1)}} + \frac{1}{\widetilde{\beta}_{ij}^{(k-1)}} \right],$$

$$\widetilde{\omega}_{ij}^{(k-1)} = \begin{cases} \widetilde{\beta}_{ik}^{(k-1)} & \text{for} \quad j > i, \quad i < k, \\ \widetilde{\beta}_{j\,k}^{(k-1)} & \text{for} \quad i > j, \quad j < k, \\ \widetilde{\beta}_{kk}^{(k-1)} & \text{for} \quad i \geq k, \quad j \geq k, \end{cases}$$

$$\widetilde{\beta}_{ij}^{(1)} = \frac{\mu_{ij} - \mu_{i1} - \mu_{1j} + \mu_{11}}{x_{i1}\, y_{j1}}.$$

It is obvious that $\widetilde{\beta}_{ij}^{(1)} = \beta_{ij}^{(1)}$. It is easy to verify that $\widetilde{\beta}_{ij}^{(k)} = \beta_{ij}^{(k)}$, $i = 2, \ldots, t_x$, $j = 2, \ldots, t_y$. Thus, in this case, relation (16) is also true.

Proposition 3. *The interpolational two-dimensional continued fractions (7) and (14) are equivalent.*

Proof. Let b_{ij}, $i = 0, \ldots, n_x$, $j = 0, \ldots, n_y$, $i \neq j$, and b_{kk}, $k = 0, \ldots, n$, be the coefficients of the interpolational two-dimensional continued fraction (7) and let b_{ij}^*, $i = 0, \ldots, n_x$, $j = 0, \ldots, n_y$, $i \neq j$, and b_{kk}^*, $k = 0, \ldots, n$, be the coefficients of the interpolational two-dimensional continued fraction (14). It is easy to see that

$$b_{00}^* = b_{00}, \quad b_{10}^* = \frac{1}{b_{10}}, \quad b_{01}^* = \frac{1}{b_{01}}, \quad b_{i0}^* = \frac{1}{b_{i0} b_{i-10}}, \quad i = 2, \ldots, n_x,$$

$$b_{0i}^* = \frac{1}{b_{0i} b_{0i-1}}, \quad i = 2, \ldots, n_y,$$

$$b_{11}^* = \frac{1}{b_{11}}, \quad b_{ii}^* = \frac{1}{b_{ii} b_{i-1i-1}}, \quad i = 2, 3, \ldots, n,$$

$$b_{ki}^* = \frac{1}{b_{k-1i} b_{ki}}, \quad i = 1, \ldots, n, \quad k = i+1, \ldots, n_x,$$

$$b_{ik}^* = \frac{1}{b_{ik-1} b_{ik}}, \quad i = 1, \ldots, n, \quad k = i+1, \ldots, n_y.$$

The algorithms presented in the previous sections enable one to independently determine the coefficients of the indicated interpolational two-dimensional continued fractions in terms of the values of the function at the grid nodes.

8. Estimate for the Remainder of the Interpolational Two-Dimensional Continued C′-Fraction

Using the Bodnar method [10] (Theorems 3.14 and 3.15), one can prove the following theorem:

Theorem 5. *If the coefficients of the continued fraction* $b_0 + \mathop{\mathrm{K}}_{i=1}^{\infty} \frac{b_i}{1}$ *satisfy the conditions* $|b_0| \le 1$ *and* $|b_i| \le \alpha = t(1-t),\ 0 \le t \le \frac{1}{2},\ i = 1, 2, \ldots,$ *then the following assertions are true:*

(i) the continued fraction is convergent;

(ii) the following estimates for the rate of convergence are true:

$$|f_n - f_m| \le \begin{cases} \dfrac{n-m}{2(n+1)(m+1)} & \text{if} \quad t = \dfrac{1}{2}, \\[2ex] \dfrac{(1-2t)t^{m+1}(1-t)^{m+1}((1-t)^{n-m} - t^{n-m})}{((1-t)^{n+1} - t^{n+1})((1-t)^{m+1} - t^{m+1})} & \text{if} \quad 0 \le t < \dfrac{1}{2}; \end{cases} \tag{21}$$

(iii) for each $n = 0, 1, \ldots,$ *the convergent* f_n *satisfies the inequality* $|f_n - b_0| \le t$.

Let $Q_k^{(s)} = 1 + \mathop{\mathrm{K}}_{i=k+1}^{s} \frac{b_i}{1}$ denote the remainder of the continued fraction.

Corollary 1. *Under the conditions of Theorem 5, the following estimate is true:*

$$\left|Q_k^{(s)}\right| \ge \begin{cases} \dfrac{s-k+2}{2(s-k+1)}, & t = \dfrac{1}{2}, \\[2ex] \dfrac{(1-t)^{s-k+2} - t^{s-k+2}}{(1-t)^{s-k+1} - t^{s-k+1}}, & 0 \le t < \dfrac{1}{2}. \end{cases} \tag{22}$$

Proof. The fraction $1 + \mathop{\mathrm{K}}_{i=1}^{\infty} \frac{-t(1-t)}{1}$ is a majorant of this continued fraction. Let P_m, Q_m, and g_m denote, respectively, the numerator, denominator, and mth convergent of the majorizing continued fraction. It can be shown that $P_m = Q_{m+1} > 0$ and

$$Q_m = (1-t)^m + t(1-t)^{m-1} + \ldots + t^m, \quad m = 1, 2, \ldots. \tag{23}$$

Using the method of mathematical induction, one can easily verify that

$$\left|Q_k^{(s)}\right| \ge g_{s-k}. \tag{24}$$

Using (23) and (24) for $t = \frac{1}{2}$, we get

$$\left|Q_k^{(s)}\right| \ge g_{s-k} = \frac{Q_{s-k+1}}{Q_{s-k}} = \frac{s-k+2}{2(s-k+1)}.$$

Performing the change of variables $t = x^{-1}$ in (23), we get

$$Q_p = \frac{(x-1)^p}{x^p} + \frac{(x-1)^{p-1}}{x^p} + \ldots + \frac{1}{x^p} = \frac{(x-1)^{p+1} - 1}{x^p(x-2)}.$$

Returning to the variable t, we obtain

$$Q_p = ((1-t)^{p+1} - t^{p+1})(1-2t)^{-1}. \tag{25}$$

Taking relations (24) and (25) into account, we get

$$\left|Q_k^{(s)}\right| \geq g_{s-k} = \frac{Q_{s-k+1}}{Q_{s-k}} = \frac{(1-t)^{s-k+2} - t^{s-k+2}}{(1-t)^{s-k+1} - t^{s-k+1}}.$$

Thus, estimate (22) is true.

Theorem 6. *Suppose that the following conditions are satisfied:*

(i) for a continuous function $f(x,y)$ defined in the domain G, the interpolational two-dimensional continued C′*-fraction (14) is constructed so that its coefficients are determined by the values of the function at the grid nodes $G_{n_{xy}}$;*

(ii) the coefficients of the interpolational two-dimensional continued C′*-fraction (14) satisfy the conditions*

$$|a_{ij}| \leq \begin{cases} t_x(1-t_x) & \forall x \in [\alpha_x, \beta_x], \quad i > j, \quad i = 0, \ldots, n_x, \quad j = 0, \ldots, n_y, \\ t_y(1-t_y) & \forall y \in [\alpha_y, \beta_y], \quad i < j, \quad i = 0, \ldots, n_x, \quad j = 0, \ldots, n_y, \\ t_x + t_y & \forall (x,y) \in G, \quad i = j, \quad i = 0, 1, \ldots, n, \end{cases}$$

where $0 \leq t_x, t_y \leq \frac{1}{2}$, $a_{ij} = b_{ij}(y - y_{j-1})$, $a_{ji} = b_{ji}(x - x_{j-1})$, and $a_{ii} = b_{ii}(x - x_{i-1})(y - y_{i-1})$;

(iii) there exists a point $(x_*, y_*) \in G$, $x_* \notin X$, $y_* \notin Y$, for which the following inequalities hold: $|a_{n_x+1j}(x_*)| \leq t_x(1-t_x)$, $j = 0, \ldots, n_y$, $\left|a_{in_y+1}(y_*)\right| \leq t_y(1-t_y)$, $i = 0, 1, \ldots, n_x$, and $|a_{n+1n+1}(x_*, y_*)| \leq t_x + t_y$, where the quantities $b_{n_x+1j}(x_*)$, $b_{in_y+1}(y_*)$, and $b_{n+1n+1}(x_*, y_*)$ are determined by relations (20) for $x_{n_x+1} = x_*$ and $y_{n_y+1} = y_*$.

Then the following estimates is true:

$$\left|f(x_*, y_*) - D_{n_{xy}}(x_*, y_*)\right| \leq \sum_{m=0}^{n} \left(\frac{2^{-2(n_x-m)}(n_x - m + 1)}{n_x - m + 3} + \frac{2^{-2(n_y-m)}(n_y - m + 1)}{n_y - m + 3} \right) + 1 \tag{26}$$

for $t_x = \frac{1}{2}$ and $t_y = \frac{1}{2}$,

$$\left|f(x_*, y_*) - D_{n_{xy}}(x_*, y_*)\right|$$

$$\leq \sum_{m=0}^{n} \left(\frac{t_x^{n_x-m+1}(1-t_x)^{n_x-m+1}((1-t_x)^{n_x-m+1} - t_x^{n_x-m+1})}{(1-t_x)^{n_x-m+3} - t_x^{n_x-m+3}} + \frac{2^{-2(n_y-m)}(n_y-m+1)}{n_y-m+3} \right)$$

$$\times \frac{1}{(t_x+1/2)^m} + \frac{1}{(t_x+1/2)^n} \tag{27}$$

for $t_x \neq \frac{1}{2}$ *and* $t_y = \frac{1}{2}$,

$$\left|f(x_*, y_*) - D_{n_{xy}}(x_*, y_*)\right|$$

$$\leq \sum_{m=0}^{n} \left(\frac{t_y^{n_y-m+1}(1-t_y)^{n_y-m+1}((1-t_y)^{n_y-m+1} - t_y^{n_y-m+1})}{(1-t_y)^{n_y-m+3} - t_y^{n_y-m+3}} + \frac{2^{-2(n_x-m)}(n_x-m+1)}{n_x-m+3} \right)$$

$$\times \frac{1}{(t_y+1/2)^m} + \frac{1}{(t_y+1/2)^n} \tag{28}$$

for $t_x = \frac{1}{2}$ *and* $t_y \neq \frac{1}{2}$, *and*

$$\left|f(x_*, y_*) - D_{n_{xy}}(x_*, y_*)\right|$$

$$\leq \sum_{m=0}^{n} \left(\frac{t_x^{n_x-m+1}(1-t_x)^{n_x-m+1}((1-t_x)^{n_x-m+1} - t_x^{n_x-m+1})}{(1-t_x)^{n_x-m+3} - t_x^{n_x-m+3}} \right.$$

$$\left. + \frac{t_y^{n_y-m+1}(1-t_y)^{n_y-m+1}((1-t_y)^{n_y-m+1} - t_y^{n_y-m+1})}{(1-t_y)^{n_y-m+3} - t_y^{n_y-m+3}} \right)$$

$$\times \frac{1}{(t_x+t_y)^m} + \frac{1}{(t_x+t_y)^n} \tag{29}$$

for $t_x \neq \frac{1}{2}$ *and* $t_y \neq \frac{1}{2}$.

Proof. Since $x_* \notin X$ and $y_* \notin Y$, we construct the interpolational two-dimensional continued C′-fraction on the basis of the values of the function $f(x, y)$ at the grid nodes $G_{n_{xy}+1} = \{x_0, \ldots, x_{n_x}, x_{n_x+1}\} \times \{y_0, \ldots, y_{n_y}, y_{n_y+1}\}$, where $x_{n_x+1} = x_*$ and $y_{n_y+1} = y_*$, as follows:

$$D_{n_{xy}+1}(x, y) = b_{00} + \Phi_0^{(n_x+1, n_y+1)}(x, y) + \mathop{\mathrm{K}}_{i=1}^{n+1} \frac{b_{ii}(x - x_{i-1})(y - y_{i-1})}{1 + \Phi_i^{(n_x+1, n_y+1)}(x, y)}, \tag{30}$$

where

$$\Phi_i^{(n_x+1,n_y+1)}(x,y) = \mathop{\mathrm{K}}_{j=i+1}^{n_x+1} \frac{b_{ji}(x-x_{j-1})}{1} + \mathop{\mathrm{K}}_{j=i+1}^{n_y+1} \frac{b_{ij}(y-y_{j-1})}{1}.$$

The two-dimensional continued C′-fraction (30) is an interpolational one, i.e., by construction, $D_{n_{xy}+1}(x_*, y_*) = f(x_*, y_*)$. Then

$$f(x_*, y_*) - D_{n_{xy}}(x_*, y_*) = D_{n_{xy}+1}(x_*, y_*) - D_{n_{xy}}(x_*, y_*).$$

The difference of $D_{n_{xy}+1}(x_*, y_*)$ and $D_{n_{xy}}(x_*, y_*)$ is determined by relation (4).

Using Theorem 5 and the method of complete mathematical induction, we prove that

$$\left|Q_k^{n_{xy}}\right| \geq t_x + t_y, \quad k = 1, 2, \ldots, n,$$
$$\left|Q_k^{n_{xy}+1}\right| \geq t_x + t_y, \quad k = 1, 2, \ldots, n+1. \qquad (31)$$

The moduli of the denominators $Q_{n_x,m}$, $Q_{n_x+1,m}$, Q_{m,n_y}, and Q_{m,n_y+1} of the continued fractions are estimated according to Corollary 1. Using estimates (22), (31), and (4), we obtain inequalities (26) – (29).

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